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HYDRAULICS OF WELLS

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Introduction

Water that is found below land surface is termed *subsurface water*. Its subsurface occurrence may be divided into zones of aeration and saturation. In the zone of aeration, the pore space of the earth is filled partly with water and partly with air. In the zone of saturation, all the interconnected pores are completely occupied by water under hydrostatic pressure. In general, the zone of aeration overlies the zone of saturation and extends upward to land surface. The saturated zone generally rests on impervious or semipervious strata. It is bounded at the top by either impervious or semipervious overlying strata or, in the absence of these strata, by what is called the *surface of atmospheric pressure*; that is, the *water table*. Actually saturation in this instance extends slightly above the water table because of capillary attraction; however, water is held here at less than atmospheric pressure.

Ground water, as distinguished from other subsurface waters, is defined as that body of water which occurs in the saturated zone and whose motion is exclusively (or, virtually) determined by gravity and by the frictional forces provoked by the motion itself. Subsurface water in the aerated zone, where motion is largely affected by capillary forces, is not included in this definition.

Wells and springs have been and will continue to be important devices for extracting ground water, whether for irrigation, industrial, or domestic uses. The importance of wells is not limited to water-supply problems. Wells have been used effectively in draining agricultural lands, controlling salt-water encroachment, relieving pressures under dams or levees, recharging ground-water basins, and disposing of radioactive wastes. The basis for improvement in well use and design is a long history of practical experience and relatively recent development of theory.

Because of the importance of well hydraulics to water economy in arid and semiarid regions of the world, the study of this branch of science has occupied the attention of many capable investigators. To obtain solutions for a ground-water problem, assumptions regarding type of flow and boundary conditions have to be made. Although such solutions often only approximate field conditions, they appear to yield results that, in many instances, have been confirmed by laboratory and field observations, and they provide valuable insight into the intricacies of ground-water flow. Notwithstanding the many simplified assumptions on which the theoretical and experimental solutions are based, well hydraulics has contributed much toward a better and more economic development of ground-water resources.

I. Basic Principles and Fundamental Equations

A. Aquifers

A geological formation, a part of a formation, or a group of formations that yields significant quantities of water is termed an *aquifer*. In contrast, an *aquifuge* is a formation containing no interconnected pores and which, therefore, can neither absorb nor transmit water. A formation which, although porous and containing water, is not capable of transmitting water in significant quantities is called an *aquiclude*.

1. Types of Aquifers

Figure 1 shows a generalized sectional view combining different types of aquifers.

Artesian aquifers, also known as *confined* or *pressure aquifers*, are those in which ground water is confined under pressure by impervious or semipervious strata. Water in a well penetrating an artesian aquifer will rise above the base of the upper confining layer; it may or may not reach land surface. Water levels in wells penetrating an artesian aquifer are the hydrostatic pressure levels of the water in the aquifer at the well sites. The imaginary surface defined by these levels is called the *piezometric surface* of the artesian aquifer. Rises and falls of water in artesian wells result primarily from changes in pressure rather than changes in storage volume.

Water-table aquifers, also known as *free*, *phreatic*, or *unconfined aquifers*, are those in which the upper surface of the zone of saturation is under atmospheric pressure. This surface is called the *water table*. Water in a well penetrating a water-table aquifer does not rise above the water table. A special case of such aquifers is the *perched aquifer*. This occurs wherever a relatively small, impervious or semipervious stratum supports a ground-water body that is above the main water table; if the bottom of the supporting layer penetrates the main ground-water body, the elevated aquifer is called a *semiperched aquifer*. Rises and falls in the water table correspond primarily to changes in the volume of water in storage within the aquifer.

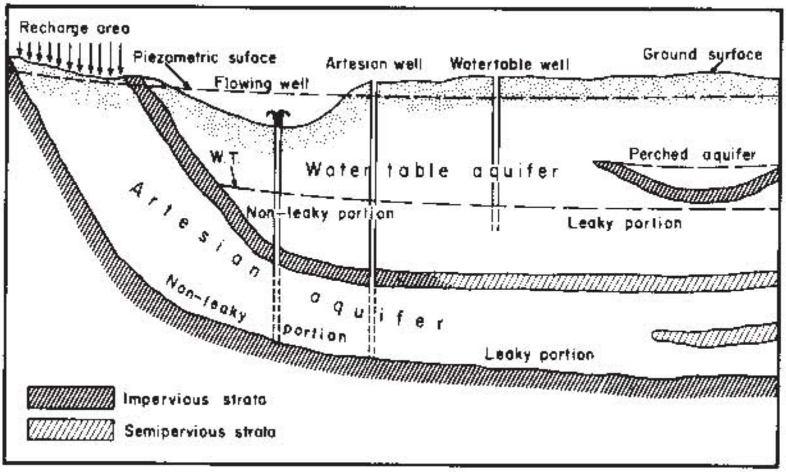

FIG. 1. Types of aquifers.

Aquifers, whether artesian or water-table, that lose or gain water through adjacent semipervious layers are called *leaky aquifers*. A water-table aquifer resting on a semipervious layer that permits slow movement of water is called a *leaky water-table aquifer*. An artesian aquifer that has at least one semipervious confining bed is called a *leaky artesian aquifer*.

Beds overlain by water-table aquifers and layers confining artesian aquifers are rarely completely impermeable. In many instances, however, the flow across the confining beds is so slow that it may be assumed negligible. Under such conditions, the aquifers are called *nonleaky aquifers*.

2. *Flow in Aquifers*

Ground water in its natural state is invariably moving. The water moves through the interconnected portion of the pore space of a saturated material open to the flow. On the scale of the pore size, the velocity of a particle of water is invariably random. The velocity changes continuously as the small section of the flow repeatedly widens and narrows and also branches and rejoins as the fluid moves through the rock or soil. This picture of random particle motion has to be reconciled with the fact that a transfer of a bulk of particles of water from a region of higher to that of lower water levels is nevertheless observed. Consider any elemental cross-sectional area cutting a great number of pores in the path of flow. Though it is not possible to say which way a particle of water will move in a given interval of time, it can be said that on the average a definite volume of water will cross the section in the direction of low water levels. Thus, on a macroscopic scale, the flow per unit area may be assumed uniform.

The water level in a piezometer (a small tube open at the bottom only) penetrating a saturated soil in which water is moving is termed the *head of the flow* at the point where the piezometer ends in the soil. The *head* (called also *piezometric* or *hydraulic head*) consists of the pressure head or the height of the column of water in the piezometer above its bottom and the elevation of the bottom of the piezometer above an arbitrarily chosen datum of elevation. Thus if p is the hydrostatic pressure at a point whose vertical coordinate is z, the head of the flow $\varphi(x,y,z,t)$ at the point (x,y,z) is defined by

$$\varphi(x,y,z,t) = \frac{p}{\gamma} + z + f \qquad (1)$$

where γ is unit weight of water, x,y,z are the rectangular coordinates, f is the elevation of the xy-plane above an arbitrarily chosen datum of elevation, and t is time of observation since an arbitrarily chosen reference of time.

On a microscopic scale, the head is unevenly distributed and its gradient along any typical stream line changes continuously as the small section of the flow repeatedly changes. However, if the head is averaged at different points over small volumes containing a great number of pores, the gradient of this average head will be sensibly uniform.

By assuming a uniform average velocity in the direction of decreasing uniform average head over an elemental soil volume, the physical system of flow is replaced by a mathematical continuum. The macroscopic or bulk velocity is a hypothetical velocity. It is not the actual or field velocity of ground water percolating through the water-bearing material. The actual velocity, known as the *effective velocity*, is measured by the volume of water passing through a unit cross section per unit time divided by the effective porosity of the soil. The *effective porosity* is the portion of pore space in a saturated permeable material in which flow of water takes place. The definition stems from the fact that not all the pore space of a material filled with water is open for the flow, since part of the voids of this material is filled with water held in place by molecular and surface-tension forces. The effective porosity of a porous medium should not be confused with its porosity. The latter is defined as the percentage of total volume occupied by the interconnected pore space of a given soil.

3. Compressibility and Elasticity of Aquifers

The concept of volume elasticity of aquifers fostered by Meinzer [1] has long been established by laboratory and field observations. Phenomena such as fluctuations of water levels in wells in response to barometric pressure changes, earthquake effects and tidal fluctuations, and surface subsidence around many wells constitute good evidence of the compressibility and elasticity (although this may be imperfect) of aquifers.

Relief of hydrostatic pressure in an aquifer due to pumping or discharging a flowing well reduces the water pressure against the confining strata. The weight of the overlying formations and other external live loads such as those due to atmospheric and tidal fluctuations was originally borne by the contained water and the solid skeleton of the aquifer. If the aquifer is compressible and elastic, reduction of water pressure (downward vertical forces remaining uniform) in the aquifer results in increasing the load borne by the aquifer skeleton. The aquifer is thus compacted, somewhat reducing the pore space. Concurrently, the water expands to the extent permitted by its elasticity. These two processes furnish part or the entire volume of water discharged by the well. After well production stops, the water pressure increases gradually, accompanied by compression of water and expansion of aquifer, thus gradually furnishing space for storing the water moving into the part of the aquifer that was affected by the compression. If the aquifer is perfectly elastic and the water levels in the intake and/or discharge areas did not change, the original hydrostatic pressure and the original volume of the aquifer will ultimately be restored, storing in the process an amount of water equal to that previously discharged.

The compressibility of an aquifer is relatively important only when the aquifer is confined and completely saturated with water. If the aquifer is unconfined, the compressibility of aquifer and water is relatively unimportant compared to changes of the water volume accompanying vertical displacement of the water table under unsteady conditions.

The relation between lowering the head and the volume of water released from storage in an aquifer because of water expansion and aquifer compression is obtained as follows:

Consider an elemental volume, $\delta V = \delta A \, \delta z$, in a compressible aquifer. The weight of the atmosphere and the weight of the material of a vertical column of cross-sectional area δA overlying this elemental volume are in equilibrium with the water pressure p and the compressive stress c_s of the solid skeleton of the aquifer, provided that the arching action of the overlying material is neglected. If the water pressure is assumed to act effectively throughout the elemental volume and the compressive stress of the solid skeleton of the aquifer is considered to act over the entire horizontal cross-sectional area δA, and if the atmospheric pressure is assumed constant, this vertical strain problem may be written as $p + c_s =$ constant, from which by differentiation

$$dc_s = -dp \tag{2}$$

Thus a decrease of water pressure is accompanied by a corresponding increase of solid compressive stress, neglecting, of course, the effects of the insignificantly small changes of the weight of water contained in the overlying column of material accompanying compression of aquifer.

If the solid skeleton of δV is compressed, its porosity θ and its vertical dimension δz will concurrently change; it is assumed that the compressive force acts in a vertical direction over a large areal extent so that changes in the lateral (horizontal) direction are negligible.

When δV is compressed, δz, according to Young's modulus of elasticity, varies with c_s in accordance with

$$d(\delta z)/\delta z = -\sigma\, dc_s = \sigma\, dp \tag{3}$$

in which the equivalent of dc_s from Eq. (2) is used, and where σ is the vertical compressibility of the solid skeleton of the material or the reciprocal of its modulus of elasticity, the negative sign indicating decrease in δz with increasing stress.

The volume δV_s of the solid material in δV is given by $\delta V_s = (1 - \theta)\,\delta V = (1 - \theta)\,\delta A\,\delta z$. As the compressibility of the individual grains of the solid material is small compared to the change in θ, the volume of the solid material may be assumed constant. Hence, $\delta V_s = (1 - \theta)\,\delta z\,\delta A =$ constant, from which by differentiation (δA is constant) there results $d(\delta V_s)/\delta A = (1 - \theta) d(\delta z) - \delta z d\theta = 0$. Using Eq. (3) and simplifying

$$d\theta = \sigma(1 - \theta)\, dp \tag{4}$$

From the definition of the bulk modulus of elasticity, the density of the water ρ increases as p increases in accordance with

$$d\rho = \beta\rho_0\, dp \tag{5}$$

where β is the compressibility of water or the reciprocal of its bulk modulus of elasticity, and ρ_0 is a reference density conveniently taken at atmospheric pressure.

The mass δM of the volume of the water δV_w contained in the elemental volume δV is given by $\delta M = \rho\delta V_w = \rho\theta\,\delta z\,\delta A$, which when δV is compressed, will change with changes in ρ, θ, and δz, δA remaining constant. Thus by differentiation, one obtains $d(\delta M)/\delta V = \theta\, d\rho + \rho\, d\theta + \rho\theta\, d(\delta z)/\delta z$, in which, using the equivalents of $d(\delta z)/\delta z$, $d\theta$, and $d\rho$ from Eqs. (2), (3), and (4), respectively, the result after simplification will be $d(\delta M)/\delta V = (\theta\beta\rho_0 + \sigma\rho)\, dp$, from which, if the small differences between ρ and ρ_0 are neglected, the change in the volume of water (water released from storage) per unit bulk volume will be given by

$$d(\delta M)/\rho\, \delta V = d(\delta V_w)/\delta V = (\theta\beta + \sigma)\, dp \tag{6}$$

From Eq. (1), it is seen that the pressure head p/γ in the elemental volume δV changes directly with the head φ; namely, $dp = \gamma\, d\varphi$ which, when substituted in Eq. (6), the result is

$$d(\delta M)/\rho\, \delta V = d(\delta V_w)/\delta V = S_s\, d\varphi \tag{7}$$

in which

$$S_s = \gamma\theta\beta(1 + \sigma/\theta\beta) \tag{8}$$

a. SPECIFIC STORAGE. The coefficient S_s, of dimension L^{-1}, has been termed *the specific storage of the aquifer* [2]. From Eq. (7) it may be defined as the volume of water which a unit volume of the aquifer releases from storage because of expansion of water and compression of the aquifer under a unit decline in the average head within the unit volume of the aquifer.

The factor $\gamma\theta\beta$ of Eq. (8) gives the fraction of storage derived from expansion of water and the product $\gamma\sigma$ gives the fraction derived from compression of aquifer.

b. STORAGE COEFFICIENT. The *storage coefficient* of an aquifer is defined as the volume of water that a vertical column of the aquifer of unit cross-sectional area releases from storage as the average head within this column declines a unit distance. In artesian aquifers where water released from or taken into storage is entirely due to compressibility of aquifer and of water, the storage coefficient S is given by $S = bS_s$, where b is the thickness of the aquifer. In water-table aquifers, the volume of water released from or taken into storage, in response to a change in head, is due mostly to dewatering or refilling the zone through which the water table moves, and partly due to water and aquifer compressibility in the saturated zone. The storage coefficient S_w for a water-table aquifer is given [3] by $S_w = \epsilon + DS_s$ where D is the height of water table above the base of the free aquifer, and ϵ is the specific yield of the aquifer. Usually, $\epsilon >>> DS_s$. Thus, S_w may, for all practical purposes, be regarded as the specific yield. The *specific yield* is defined as the ratio of the volume of water that a rock or soil will yield by gravity to its own volume. In other words, it represents very closely the effective porosity.

Example 1. The observed barometric efficiency of an artesian aquifer is 0.60. If the uniform thickness of the aquifer is 150 ft and its average porosity is 0.31, estimate the storage coefficient of the aquifer.

Because of the elasticity of artesian aquifers, water levels in artesian wells fluctuate with changes in atmospheric pressure. The *barometric efficiency BE* is defined as the ratio of the net change in water level dh observed in an artesian well to the corresponding net change $d(p_a/\gamma)$ in atmospheric pressure p_a expressed in feet of water; that is, $BE = dh/(dp_a/\gamma)$. For a variable atmospheric pressure, the relation just preceding Eq. (2) becomes $p + c_s = p_a +$ constant; whence, $dp - dp_a = - dc_s$. The weight of the column of water in the well above the top of the aquifer plus that of the atmosphere is balanced by the water pressure in the aquifer; hence, the change of water level in the well is given by $\gamma\, dh = dp - dp_a$. Consequently, $BE = \gamma\, dh/dp_a = - dc_s/(dp + dc_s) = -1/(1 + dp/dc_s)$. Now, if the sand grains are assumed incompressible, then as the aquifer is compressed the change in the bulk volume $d(\delta V)$ is equal to the change of water volume $d(\delta V_w)$. Thus, since $\delta V_w = \theta\, \delta V$ and $d(\delta V_w) = d(\delta V)$, then $d(\delta V_w)/\delta V_w = d(\delta V)/\theta \delta V$. From the definition of bulk modulus of elasticity, $d(\delta V_w)/\delta V_w = - \beta dp$ and $d(\delta V)/\delta V = -\sigma\, dc_s$, negative sign indicating a decrease in volume accompanying an increase in stress. Consequently, $dp/dc_s = \sigma/\theta\beta$; hence $BE = -1(1 + \sigma/\theta\beta)$; the negative sign signifying that a rise in barometric pressure is accompanied by depression of water level in the well. If in this BE relation, Eq. (8) and $S_s = S/b$ are substituted, then $S = \gamma\theta\beta b/BE$ from which with $\theta = 0.31$, $\gamma = 0.0361$ lb/in.^{3}, $\beta = 3.3 \times 10^{-6}$ in.^{2}/lb, $b = 1800$ in., and $BE = 0.60$, the storage coefficient S is computed by a slide rule as 1.1×10^{-4}.

Example 2. If the yield of a well field draining the artesian aquifer of Example 1 is entirely from storage, find the percentage of yield that is due to aquifer compression.

The total production of the well field is proportional to S. From Eq. (8), the storage coefficient is given by $S = \gamma\theta\beta b + \gamma b\sigma$, or $1 = (\gamma\theta\beta b/S) + (\gamma b\sigma/S)$. Thus, the first term represents percentage owing to water expansion and the second owing to aquifer compression. So the percentage of yield owing to aquifer compression $= 1 - \gamma\theta\beta b/S = 1 - BE = 1 - 0.60 = 0.40$, or 40%.

Example 3. In response to a uniform tide in an effectively long and fairly straight channel that cuts completely through a semi-infinite sand, the water levels in an aquifer fluctuate with an amplitude $A = h_0 \exp(-x\sqrt{\pi S/t_0 T})$, where h_0, t_0 are, respectively, the amplitude and period of uniform tide, S and T are, respectively, the storage and transmissivity coefficients of the aquifer, and x is the distance from the channel of uniform tide to any point in the aquifer. If this aquifer overlies an artesian aquifer which has no direct contact with the source of uniform tide, find the amplitude of water-level fluctuation in the deeper aquifer.

The *tidal efficiency*, *TE*, of an aquifer is defined as the ratio of the change of the water level $dh(=dp/\gamma)$ in a well tapping the aquifer to the corresponding change in the stage of the tide dH in the overlying strata (corrected for density, if necessary). Thus, $TE = dh/dH = dp/\gamma\, dH$. Since the vertical load on the aquifer is no longer constant but changes as the stage of the tide, the relation preceding Eq. (2) may then be written as $p + c_s = \gamma H +$ constant, where the top of the deeper aquifer is taken as the datum of elevation. Consequently, $\gamma\, dH = dp + dc_s$; hence, $TE = (dp/dc_s)/(1 + dp/dc_s)$, from which, since (from Example 1) $dp/dc_s = \sigma/\theta\beta$, $TE = (\sigma/\theta\beta)/(1 + \sigma/\theta\beta)$. Now numerically $BE = 1/(1 + \sigma/\theta\beta)$; hence, $BE + TE = 1$. Thus $TE = 1 - BE = 1 - \gamma\theta\beta b/S$. From definition of TE, the amplitude of fluctuation in the deeper aquifer is $A' = ATE$, or $A' = (1 - \gamma\theta\beta b/S)A$.

It is of interest to observe that for a perfectly elastic aquifer, $TE = 1 - BE$, and that the percentage of storage attributable to expansion of water is equal to BE and that attributable to compression of aquifer is equal to TE (Example 2.)

B. Differential Form of Darcy's Law

In laminar flow through porous media, the inertial forces that are associated with even rapid changes of flow rates are so much smaller than the viscous forces that they can be neglected in nearly all problems of practical interest [4]. If on a macroscopic scale the ground-water flow is laminar (flow is orderly or the bulk of particles moves substantially in parallel paths), the only macroscopic force exerted on the water by the solid skeleton of the medium of flow is that associated with the viscous resistance to flow. This force must, therefore, be in equilibrium with the weight of the water and the net force due to water pressure in the direction of flow.

Consider an elemental volume $\delta V = \delta A\ \delta s$ of the aquifer arbitrarily oriented in the field of flow, where δA and δs are, respectively, the elemental area and the normal elemental length of the volume. If the water pressure acting on one end of the elemental volume is p, then the pressure on the other end is $p + (\partial p/\partial s)\,\delta s$. Consequently, the net force f_p in the direction of δs is $f_p = -\theta\delta A(\partial p/\partial s)\,\delta s$.

The weight of water in δV is equal to $\gamma\theta\ \delta A\ \delta s$. The component of this force f_g parellel to δs is $f_g = -\gamma\theta\ \delta A\ \delta s(\partial z/\partial s)$, where $(\partial z/\partial s)$ is the sine of the angle that δs makes with its projection on the horizontal plane.

Assuming that the total surface of the grains within δV is proportional to $\delta A\ \delta s$, then if the dynamic viscosity of water is μ, and v_s is the bulk velocity in the direction of δs, the viscous force opposing the flow and acting parallel to δs may be given by $f_\mu = -\mu a_s\ \delta A\ \delta s v_s$, where a_s is a constant characteristic of the geometry of the pores in the direction of δs and proportional to the specific surface.

The forces f_p, f_g, and f_μ are in equilibrium. Thus

$$-[\partial p/\partial s + \gamma\ \partial z/\partial s + (\mu a_s/\theta)v_s]\theta\,\delta V = 0$$

or

$$v_s = -(\gamma k_s/\mu)\ \partial(p/\gamma + z)/\partial s \tag{9}$$

where k_s, a constant of dimension L^2, is substituted for θ/a_s and is characteristic of the aquifer. Dimensional analysis shows that it varies with the square of the mean grain diameter and with a dimensionless constant that depends upon the porosity, the range and distribution of grain size, the shape of the grains and their orientation and arrangement, all of which are characterized by dimensionless ratios and angles. This coefficient is appropriately called the *permeability of the aquifer* in the direction of v_s, as it characterizes the ease with which water can permeate the aquifer in that direction.

From Eq. (1), $\varphi = p/\gamma + z + f$; hence, $\partial\varphi/\partial s = \partial(p/\gamma + z)/\partial s$, since f is a constant. In terms of the head φ, Eq. (9) becomes

$$v_s = - K_s \, \partial\varphi/\partial s \tag{10}$$

in which $K_s = (\gamma/\mu)k_s$ and is known as the *hydraulic conductivity* or simply the *conductivity* of the aquifer in the direction of v_s.

Equation (10) is the differential form of Darcy's law. The formula states that the flow rate through porous material in any direction is proportional to the negative rate of change of the head in that direction. The negative sign signifies that the fluid moves in the direction of decreasing head. The rectangular-coordinate bulk velocities may, therefore, be written as

$$v_x = - K_x \, \partial\varphi/\partial x, \qquad v_y = - K_y \, \partial\varphi/\partial y, \qquad v_z = - K_z \, \partial\varphi/\partial z \tag{11}$$

where K_x, K_y, and K_z are the conductivities in the x-, y-, and z-directions, respectively.

The vector velocity at any point in an aquifer may be taken as the vector sum of the component velocities [5].

Example 4. In Fig. 2 an artesian aquifer discharges fresh water into the sea at a rate q per unit width of sea front. A salt-water wedge exists at the intersection of the aquifer and the sea. If no flow occurs in the salt-water zone and if the piezometric surface represents the average heads along vertical sections of the aquifer, find an expression for the length of the intruded salt-water wedge.

Let the sea level be the datum of elevation. From Fig. 2, the head of the flow at any distance x from the aquifer mouth is φ. At any point A on the interface the hydrostatic pressure $\gamma_s Z$ of sea water is balanced by the total pressure $\gamma_f(Z + \varphi)$ of fresh water. Consequently, $Z = \varphi/a$, where $a = (\gamma_s - \gamma_f)/\gamma_f$. Consider the flow across a vertical section at the distance x. Its area per unit width of sea front is $Z - b' = \varphi/a - b'$. Thus, $q = (\varphi/a - b')(- v_x)$, or from Darcy's law, $q = (\varphi/a - b')K \, \partial\varphi/\partial x$ or

$$\partial(\varphi - ab')^2/\partial x = 2qa/K$$

At $x = 0$, $\varphi(0) = aZ(0) \approx ab'$ and at $x = l$, $\varphi(l) = aZ(l) = a(b + b')$. Hence, integration of the preceding equation gives

$$[(\varphi - ab')^2]_{\varphi(0)}^{\varphi(l)} = [(2qa/K)x]_0^l$$

whence

$$l = aKb^2/2q$$

Sand models at the University of California demonstrated the validity of this formula [5].

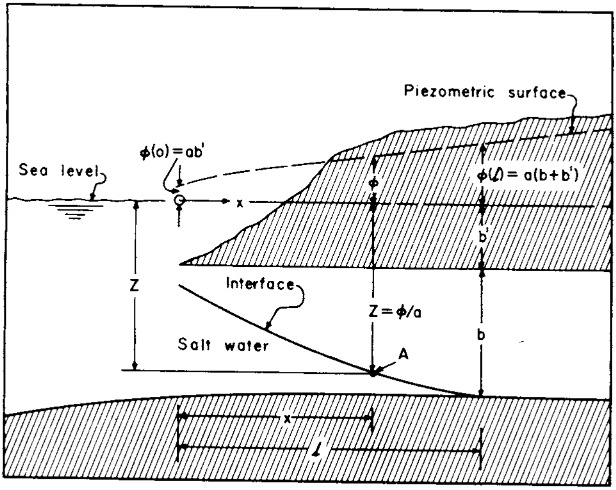

FIG. 2. Salt-water wedge in an artesian aquifer.

1. Hydraulic Conductivity

The constant of proportionality in Darcy's law is called the *coefficient of hydraulic conductivity*, or simply, the *conductivity*. It is also known as the *transmission constant*, and though inappropriately so, as the *coefficient of permeability*. The coefficient K_s depends not only on the permeability of the aquifer k_s but also on the unit weight and dynamic viscosity of the fluid. The last two parameters define important properties of the fluid. The unit weight may be considered as the driving force per unit hydraulic gradient, whereas the viscosity may be regarded as a measure of the resistive force per unit area per unit transverse velocity gradient within the pores of the aquifer. The dependence of K_s on k_s, μ, and γ makes it vary not only from aquifer to aquifer and from liquid to liquid, but also from direction to direction and from temperature to temperature. Even in homogeneous materials, K_s may vary with flow direction, in which event it is termed *anisotropic conductivity*. Anisotropic conductivity is a common occurrence in unconsolidated sedimentary deposits where the horizontal conductivity may exceed that of the vertical 2 to 10 times or more. In many ground-water problems, the temperature does not vary appreciably, and the aquifer may be assumed homogeneous and isotropic. Thus the coefficient K_s may be actually regarded as constant with temperature and direction in the equations of flow. This conductivity is generally symbolized by the letter K.

From Darcy's law, the hydraulic conductivity may be defined as the volume of water per unit time passing through a medium of unit cross-sectional area, if the magnitude of the hydraulic gradient prevailing at the section is unity. It has the dimension of velocity (L/T).

2. *Transmissivity*

In aquifers that may be assumed uniform in thickness, the product of the thickness b of the aquifer and the average value of the hydraulic conductivities in a vertical section of the aquifer is termed the *coefficient of transmissivity* or simply the *transmissivity* of the aquifer. It characterizes the ability of the aquifer to transmit water. The transmissivity of dimension L^2/time, is generally designated by T. Thus $T = Kb$.

The specific storage S_s, the storage coefficient S, the specific yield ϵ, the conductivity K, and the transmissivity T constitute, with other coefficients, what is usually referred to as the *formation coefficients*, *constants*, or *parameters*, or the *hydraulic properties of aquifers*.

Because of the many factors on which these coefficients depend, numerical values must depend on experimental determination. Although various laboratory techniques are available, more reliable results are obtained from pumping tests by which the aquifers are tested in place. These tests may be regarded as techniques for aquifer calibration.

3. *Range of Validity of Darcy's Law*

In deriving Darcy's law, the flow is assumed laminar; that is, the velocity of flow is proportional to the first power of the hydraulic gradient. This assumption has long been verified by observation [5, 6].

The range of flow rates for which laminar flow exists has been studied by numerous investigators. By analogy to flow through pipes, the Reynolds number is used as an index to establish the limit of flows corresponding to which the laminar-flow regime breaks down. The Reynolds number is given by $\mathbf{R} = \rho v d/\mu$ in which, when adapted to flow through porous media, v is the bulk velocity and d is the mean grain diameter, defined as a diameter such that, if the grains were of that diameter, the porous sand would transmit the same amount of fluid that it actually does; the mean grain diameter is determined by Hazen as the diameter of sand grain such that 10% of the natural sand (by weight) is of smaller grains and 90% is of larger grains. The symbols ρ and μ have been defined.

For very low velocities, laminar flow occurs; consequently, from both theory and experiment, no lower limit is known to exist for the applicability of Darcy's law [5, 6]. However, in material of extremely fine grains, such as colloidal clays (in which the pores may be reduced to a few molecular diameters) or sands that are not completely saturated with water, the law of flow may be somewhat different from Darcy's law.

Experiments have shown [5, 6] that departure from laminar flow begins at values of **R** between 1 and 10, depending upon the range of grain size and shape. Fortunately, almost all natural ground-water motion has Reynolds numbers less than unity, and thus Darcy's law is applicable. In rock aquifers, in unconsolidated aquifers with steep hydraulic gradients, or in aquifers containing large-diameter solution channels, the flow may be expected to deviate from Darcy's law. Moreover, flows in the immediate vicinity of open bodies of water, such as to streams and wells where they are often associated with steep gradients, may also be expected to violate Darcy's law. In the majority of these instances, however, it may be assumed that the flow remains laminar to the face of the drainage facility. This assumption is implicitly made in the treatment presented in this article.

Example 5. The slope of the piezometric surface of the aquifer of Example 4 is approximately equal to 0.0001. The aquifer, 100 ft thick, is composed of uniform sand having a mean grain diameter of 0.004 ft. A 12-in. well screened throughout the aquifer is pumping at a rate equal to 6.28 ft^3/sec. Does the flow near the well obey Darcy's law? What is the length of salt intrusion if the well is far enough inland so that it does not disturb the seaward natural flow?

The velocity at the face of the well $v = Q/2\pi r_w b = (6.28)/(2\pi)(0.5)(100) =$ 0.02 ft/sec. From Reynolds number $\mathbf{R} = vd/(\mu/\rho) = vd/\nu = (0.02)(0.004)/$ (0.00001), where $\nu = \mu/\rho$ is the kinematic viscosity $= 0.00001$ ft^2/sec for water. Thus, $\mathbf{R} = 8$. This suggests that the flow is departing from its laminar regime as it arrives at the face of the well. The well under consideration, however, is a fairly productive well. Ordinarily, the flow may be assumed to remain laminar to the face of the well.

From Example 4, $l = aKb^2/2q$. From Darcy's law, $q = KbI$; hence, $l = ab/2I$. The slope of the piezometric surface is very small so that the hydraulic gradient (sine of the angle of inclination of the piezometric surface) may be taken equal to the slope (tangent of angle of surface inclination). Thus $I =$ 0.0001. And if γ_s and γ_f are equal to 63.95 and 62.4 lb/ft^3, respectively, then $a = 0.025$ (Example 4). Consequently, $l = (0.025)(100)/2(0.0001) = 12{,}500$ ft.

4. *Ground-Water Velocity Potential*

If a vector $\bar{v}$ can be represented as the gradient (positive or negative, depending on convention) of a scalar function V, then V is called the potential of $\bar{v}$. Thus, if $\bar{v}$ is a velocity vector, V is called the *velocity potential.* Symbolically, this may be written as $\bar{v} = -\nabla V$, where ∇ is the vector differential operator with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$. Whence

$$v_x = -\partial V/\partial x, \qquad v_y = -\partial V/\partial y, \qquad \text{and} \qquad v_z = -\partial V/\partial z$$

According to Eq. (11), the head φ, a scalar function, appears to have the properties of a potential. In aquifers of constant and isotropic permeability (k_s = constant), the conductivity of the homogeneous and isotropic aquifer may be considered constant ($K_s = K$) for a given fluid at uniform temperature. Consequently, the coordinate velocities are

$$v_x = -K\, \partial\varphi/\partial x, \qquad v_y = -K\, \partial\varphi/\partial y, \qquad v_z = -K\, \partial\varphi/\partial z \tag{12}$$

Clearly, the product $K\varphi$ is a scalar function whose negative space derivative in any direction gives the velocity in that direction. The function $V = K\varphi$, by definition, is the ground-water velocity potential.

The existence of a velocity potential implies irrotational motion. This appears to contradict the fact that for Darcy's law to be applicable the flow must be viscous and hence invariably rotational. This would be true of velocities on the scale of the pore size. However, Eq. (10) and hence Eq. (12) apply to bulk velocities, or average velocities, over a macroscopic area large enough that the rotational components balance out and the resultant flow is irrotational. In other words, the potential function deals only with bulk velocities and can be applied to the viscous flow of ground water. Thus, a powerful mathematical tool becomes available for solving flow problems which otherwise are not amenable to even approximate treatment.

C. Differential Equation of Ground-Water Motion

By the law of conservation of matter, the net inward flux through an arbitrary element of volume δV situated in the field of flow plus the amount of mass generated therein per unit time must equal the rate of matter accumulating within this volume. If δM is the mass contained in the volume δV, then the rate of change of this mass per unit time per unit volume will be $[\partial(\delta M)/\partial t]/\delta V$. Let $\bar{v}$ be the bulk velocity vector and ρ be the density of the fluid, then the flux of matter transported or mass flow per unit time per unit area is $\rho\bar{v}$. If water is being released within the field of flow at the rate of $F(x,y,z,t)$ per unit time per unit volume, the mass added per unit time (this mass may be regarded as being generated within the field of flow) per unit volume is ρF. Since mass is conserved, it follows that

$$-\oint (\rho\bar{v}) \cdot dS + \iiint \rho F\, dV = \iiint [(\partial\, \delta M/\partial t)/\delta V]\, dV$$

in which the first term is the surface integral taken over the closed surface of δV and the others are volume integrals extended over the volume of the element.

Making use of the divergence theorem to change the surface integral into a volume integral, replacing the integrand of the right-hand member of the above relation with its equivalent from Eq. (7) and rearranging will yield

$$-\iiint [\nabla \cdot (\rho \bar{v}) - \rho F + \rho S_s \, \partial\varphi/\partial t] \, dV = 0$$

where ∇ is the gradient operator (Section I, B, 4). As this relation must hold for any arbitrary volume, the integrand must be zero. Consequently, after performing the vector operations, one obtains

$$-[\partial(\rho v_x)/\partial x + \partial(\rho v_y)/\partial y + \partial(\rho v_z)/\partial z] + \rho F = \rho S_s \, \partial\varphi/\partial t \tag{13}$$

Jacob [6] has expanded the bracketed terms of Eq. (13) and put the resulting expressions of the components of the density gradients in terms of the hydraulic gradient; then he showed that these terms can be neglected in comparison with other terms of the equation. As the variation of water density in space is very small, it may be considered essentially constant in most problems of interest. Thus, if ρ is constant in space, Eq. (13) can be written, through use of Eq. (11), as

$$K_x \, \partial^2\varphi/\partial x^2 + K_y \, \partial^2\varphi/\partial y^2 + K_z \, \partial^2\varphi/\partial z^2 + F = S_s \, \partial\varphi/\partial t \tag{14}$$

For an isotropic aquifer, Eq. (14) becomes

$$\nabla^2\varphi + F/K = (1/\nu) \, \partial\varphi/\partial t \tag{15}$$

in which ∇^2 is the Laplacian operator and $\nu = K/S_s = T/S$.

Equations of the same form as Eqs. (14) and (15) appear in other branches of mathematical physics, as in the theories of unsteady flow of heat and electricity and in the theory of unsteady diffusion from which solutions to many ground-water flow problems may be obtained by analogy.

D. Initial and Boundary Conditions

Solution of Eqs. (14) and (15), or approximate equations resulting therefrom, that are of interest in practical applications are those which satisfy certain initial conditions and conditions prevailing on the boundaries of a given ground-water flow system. On small scales of space and time, the actual variations of boundary and initial conditions are generally irregular. However, if averaged in time and on the boundaries, general trends and space variation may be tractable and discernible. Such average variations may be given mathematical expressions. Similarly, mathematical expressions may be given to the usually irregular shapes of the boundaries of a flow system if they are idealized by simple geometrical forms such as straight lines, exponential curves, or circular and elliptical arcs. Such simplifying assumptions make possible the mathematical treatment of many ground-water problems, yielding results which in many instances have been confirmed by observation.

1. Initial Conditions

In unsteady flow, the distribution of head throughout a flow system at a particular time, usually taken as the initial time of the problem, is assumed to be known. If it is represented by a continuous function, the solution of the problem should tend to this function as the time tends to zero. If, on the other hand, the initial distribution is discontinuous at points or surfaces, these discontinuities must disappear after a very short time, and the solution must converge to the initial distribution at all points where this distribution is continuous.

2. Boundary Conditions

The boundaries of a flow system and the conditions prevailing thereupon that are of common occurrence in the theory of ground-water motion are as follows.

a. OPEN BOUNDARIES. Pervious or semipervious boundaries such as bottoms and banks of rivers, canals, lakes, earth dams, and bodies of surface water which permit the entry or exit of water are called open boundaries. In general, the conditions on these boundaries are as follows:

Known head distribution; that is, φ (on a boundary) $= f$ (position and time). If the boundary is a body of surface water whose volume is so large that its level is uniform and independent of changes in ground-water flow, the head distribution thereupon may be considered uniform in position; it may be either constant or variable with time. Such boundaries may be treated as surfaces of equal head.

Boundaries open to the atmosphere so that ground water emerges from them to evaporate or to trickle down the boundary face may be regarded as surfaces of uniform atmospheric pressure, provided effects of surface tension and possible restrictions to flow can be neglected. A water table receiving significant amounts of water from downward percolation of surface water may also be considered as a surface of uniform atmospheric pressure, provided that the capillary effects are neglected. The stream lines under these conditions will cross the water table at various angles instead of lying in it. These angles are given by [6] $\theta = \tan^{-1}[(K/w) \tan \delta] - \delta$, where K, w, δ, and θ are, respectively, the conductivity, the rate of uniform vertical downward infiltration, the angle of inclination of the water table with the horizontal, and the angle of refraction of the stream lines with the vertical.

If the atmospheric pressure is taken as zero, the head distribution on boundaries open to the atmosphere, from definition of φ with $p = 0$, is φ (on boundary open to atmosphere) $= z$.

Known flow distribution; that is v_s (at boundary) $= -K_s\ \partial\varphi/\partial s = f$ (position and time). If by artificial control, such as that effected by a pump in a well, the total flow through a water entry section may be held constant and if the distribution of the flow is uniform over such a section, the function f may be assumed constant. Flow from or into adjacent semipervious boundaries of a flow system may be proportional to the head distribution over those boundaries. In such systems v_n (at boundary) $= -K_n\ \partial\varphi/\partial n = c_1 + c_2\varphi$ (on boundary) where c_1 and c_2 are constants depending on flow conditions, and n is the direction of a line normal to the boundary.

b. CLOSED BOUNDARIES. Impervious or nearly impervious boundaries of a ground-water body, such as under- or overlying beds or contiguous rock masses (as along a fault or the walls of buried rock valleys) or dikes or similar structures, are closed to the flow of ground water. The velocity normal to these boundaries is, therefore, equal to zero, or v_n (at boundary) $= -K_n\ \partial\varphi/\partial n = 0$.

Stream surfaces in a flow system may be considered as impervious boundaries. This is because the flow does not cross these surfaces. Consequently, the preceding boundary condition prevails thereupon.

c. INTERFACE BOUNDARIES. Frequently two adjacent aquifers have uniform but different hydraulic properties. The change may be gradual or abrupt. The zone of this change may be idealized as a surface forming a common boundary of the two aquifers. If the head distribution in two such aquifers is designated by φ_1 and φ_2 and their conductivities by K_1 and K_2, respectively, then, since one value of the head may obtain at any point, one condition is φ_1 (on interface) $= \varphi_2$ (on interface) and since what enters the boundary at one side must come out on the other side, the velocities normal to the interface must be equal on the two sides; hence, the other condition is v_{1n} (at interface) $=$ v_{2n} (at interface) or $K_1\ \partial\varphi_1/\partial n = K_2\ \partial\varphi_2/\partial n$.

From consideration of these conditions, the refraction in flow lines when the flow passes from one medium to the other is given by $K_1/K_2 = \tan\theta_1/\tan\theta_2$ where θ_1 and θ_2 are the angles the flow lines make with the normal to the boundary [5].

d. FREE-SURFACE BOUNDARY. A *free surface* is defined as a stream surface along which the pressure is uniform. In problems of unconfined flow in which the capillary effects and the rate of downward percolation of surface water and its accretion to the water table may be neglected, the water table may be regarded as the upper bounding surface of the flow. The water table under these conditions is a free surface; the uniform pressure on it is that of the atmosphere. Hence, the head, from Eq. (1), on a free surface is φ (on free surface) $= \varphi(x, y, D, t) = p_a/\gamma + D + f = D + f$ where p_a is the atmospheric pressure (taken as zero) and D is the vertical coordinate of any point on the free surface. The differential equation of the free surface in an isotropic aquifer is obtained as follows:

Since $\varphi(\bar{R},t) = p/\gamma + z + f$, where $\bar{R}$ is the vector locating a point in the field of flow, the total change in p with time is

$$d(p/\gamma)/dt = d[\varphi(\bar{R},t) - z - f]/dt$$
$$= \partial\varphi/\partial t + \nabla\varphi \cdot d\bar{R}/dt - dz/dt$$

but

$$d\bar{R}/dt = \bar{v}_e = \bar{v}/\epsilon = -(K/\epsilon)\nabla\varphi$$

and

$$dz/dt = v_{ez} = v_z/\epsilon = -(K/\epsilon)\ \partial\varphi/\partial z$$

in which the subscript e pertains to "effective velocities" or "seepage velocities." Whence, at the free surface where $dp/dt = 0$, the result after rearranging is

$$(\epsilon/K)\ \partial\varphi/\partial t = \nabla\varphi \cdot \nabla\varphi - \partial\varphi/\partial z$$

in which z is to be replaced by D (the vertical coordinate of any point on the free surface) after the differentiation is performed.

Example 6. The steady-state head distribution in a thick earth dam with zero outflow head may be approximated by $\varphi(x,z) = a[x + \sqrt{x^2 + z^2}]^{0.5}$, where $a = \sqrt{q/K}$, q is the rate of seepage per unit width of dam and x and z are the rectangular coordinates with origin at the intersection of the base and outflow face of the dam [7]. Find the equation of the free surface.

At the free surface, $z = D$; D being the height of free surface above the base of the dam. If this base is taken as the datum of elevation, f of Eq. (1) is zero; hence, $\varphi(x,D) = D$. Substituting in the expression of $\varphi(x,z)$ yields $D = a[x + \sqrt{x^2 + D^2}]^{0.5}$. By solving for D, the free-surface equation is $D^2 = 2a^2x + a^4$.

E. Approximate Differential Equation of Motion

The solution of Eqs. (14) and (15) that satisfies the conditions specified on the boundaries of a given flow system will give the distribution of head throughout the system, as well as on its boundaries. It is quite difficult, however, to solve this problem exactly in the general case. Instead, examination should be made of the possibility of describing the flow in systems of general interest by an approximation of Eqs. (14) and (15) in problems for which rigorous solutions of these equations are either formidable to use or difficult to obtain. Many flow problems in which the head varies in the vertical direction, such as the flow in water-table aquifers, leaky aquifers, or in aquifers of nonuniform thickness, may be treated in terms of a head distribution that is averaged in the vertical direction. This process not only makes the solution of problems easier to obtain but also in many instances desirable, as water levels observed in the field generally represent average heads in vertical sections of the aquifer. In wells, water levels represent the average head on the water entry section of the well. On the lateral boundaries of a flow system, such as lakes, streams, or seas which cut or may be assumed to cut completely through the aquifer, water levels represent the average head on the vertical section of the boundary.

In Fig. 3, ground water flows between the surfaces $z_1 = f(x,y)$ and $z_2 = H(x,y,t)$. The main flow loses and/or gains water from the adjacent layers. In many instances, the flow in the over- and underlying layers may be considered as nearly vertical, such as when downward percolation feeds a water table

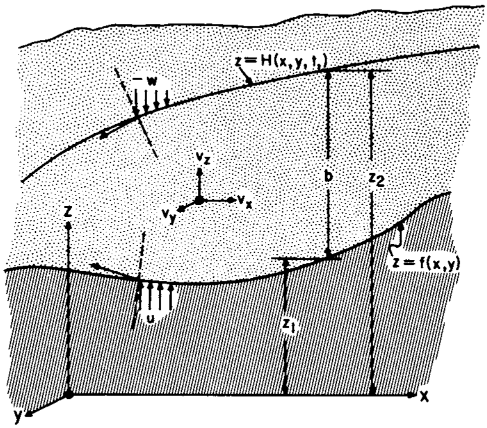

FIG. 3. Diagrammatic representation of flow in a vertically replenished aquifer.

or when vertical leakage through semipervious layers occurs. Let the rate at which vertical leakage crosses the interface f per unit area be u and that which crosses the surface H be w. If u and w are zero, the surface f and H are stream surfaces. Consequently, differentiation following a particle of water on these surfaces results, respectively, in

$$dz_1/dt = (\partial f/\partial x)(dx/dt) + (\partial f/\partial y)(dy/dt)$$

and

$$dz_2/dt = (\partial H/\partial x)(dx/dt) + (\partial H/\partial y)(dy/dt) + \partial H/\partial t$$

but dl/dt represents the effective (seepage) velocity in the direction of l; that is, v_l/ϵ, where v_l is the bulk velocity in the l-direction and ϵ is the effective porosity (specific yield) of the material. Expressing the total derivatives of these relations in terms of bulk velocity components yields

$$v_{1z}(f) = v_x(f)(\partial f/\partial x) + v_y(f)(\partial f/\partial y)$$

and

$$v_{2z}(H) = v_x(H)(\partial H/\partial x) + v_y(H)(\partial H/\partial y) + \epsilon\partial H/\partial t$$

where, for purposes of simplification, $v(H)$ and $v(f)$, respectively, denote $v(x,y,H,t)$ and $v(x,y,f,t)$.

If u and w are not zero, the vertical component of the velocity at the surface f will be given by

$$v_z(f) = u + v_{1z}(f)$$

and that at the surface H by

$$v_z(H) = w + v_{2z}(H)$$

where v_z, u, and w are taken positive in the positive direction of z.

The equation of motion in the main aquifer for a water of constant density (ρ = constant) can be written from Eq. (13) as

$$\partial v_x/\partial x + \partial v_y/\partial y + \partial v_z/\partial z - F = -S_s\, \partial\varphi/\partial t$$

which, when integrated with respect to z between the limits $f(x,y)$ and $H(x,y,t)$ and the differentiation and integration operations are then interchanged through use of the rule of differentiation under the integral sign, will become

$$\begin{aligned} &\partial\Big[\int_f^H v_x\, dz\Big]/\partial x - v_x(H)\, \partial H/\partial x + v_x(f)\, \partial f/\partial x \\ &+ \partial\Big[\int_f^H v_y\, dz\Big]/\partial y - v_y(H)\, \partial H/\partial y + v_y(f)\, \partial f/\partial y \\ &+ v_z(H) - v_z(f) - \int_f^H F\, dz \\ = &- S_s\Big\{\partial\Big[\int_f^H \varphi\, dz\Big]/\partial t - \varphi(H)\, \partial H/\partial t\Big\} \end{aligned}$$

where $\varphi(H)$ is used to denote $\varphi(x,y,H,t)$.

If, in the preceding equation, $v_z(H)$ is replaced by its equivalent in terms of w and $v_{2z}(H)$, and $v_z(f)$ in terms of u and $v_{1z}(f)$, and v_x and v_y are replaced by their equivalents from Eq. (12), and if the rule of differentiation under the integral sign is used again to interchange the differentiation and integration operations, the result will be

$$\begin{aligned} &(\partial/\partial x)[\partial(b\bar{\varphi})/\partial x - \varphi(H)\, \partial H/\partial x + \varphi(f)\, \partial f/\partial x] \\ &+ (\partial/\partial y)[\partial(b\bar{\varphi})/\partial y - \varphi(H)\, \partial H/\partial y + \varphi(f)\, \partial f/\partial y] \\ &+ (u - w)/K + b\bar{F}/K = (\epsilon/K)\, \partial H/\partial t \\ &+ (S_s/K)[\partial(b\bar{\varphi})/\partial t - \varphi(H)\, \partial H/\partial t] \end{aligned} \tag{16}$$

where $\varphi(f) = \varphi(x,y,f,t)$, $\varphi(H) = \varphi(x,y,H,t)$, and $\bar{F}(x,y,t)$ and $\bar{\varphi}(x,y,t)$ are, respectively, the average rate at which water is added (hypothetically generated) per unit time per unit volume to the flow and the average head over the thickness $b = H - f$; that is,

$$\bar{F} = (1/b) \int_f^H F\, dz, \qquad \text{and} \qquad \bar{\varphi} = (1/b) \int_f^H \varphi\, dz$$

Depending on the nature of variation of H, f, u, and w, Eq. (16) may be simplified considerably for many problems of practical interest (Sections III and V).

II. Integral Transforms and Mathematical Functions

The theory of integral transforms affords mathematical devices through which solutions of various problems in mathematical physics can be obtained. The Laplace and the finite Fourier transformations are two such devices. In problems of ground-water flow, the first transformation removes the time variable and the second removes after each application one of the space variables, provided the region is finite in the direction of that space variable. Full account of these transformations and their applications may be found in the mathematical literature. The concern here is with the application to ground-water problems, the aim being to describe rather than to justify the procedure. Fundamental properties of these transformations will be given with neither a proof nor a careful statement of the conditions under which they apply. The exact conditions are, in fact, not needed since the final solution of a problem can be verified by applying the boundary conditions and by direct substitution in the differential equation.

A. Laplace Transforms

Unless it is necessary to emphasize the dependence of the functions $s(x,y,z,t)$ and $\bar{s}(x,y,z,p)$ on (x,y,z), they will be denoted henceforth by s and $\bar{s}$, respectively. Should it be necessary to emphasize the dependence of s on t and $\bar{s}$ on p, they will be written as $s(t)$ and $\bar{s}(p)$, respectively.

Given a function $s(t)$ defined for positive values of t, its Laplace transform $\bar{s}(p)$ is defined by

$$L\{s(t)\} = \bar{s}(p) = \int_0^\infty \exp(-pt)s(t)\,dt$$

where p is a number, called the *transform variable*, whose real part is positive and large enough to make the integral convergent.

The inverse transformation, symbolized by $L^{-1}\{\bar{s}(p)\} = s(t)$, is defined by a contour integral (not given here) in the complex plane. In this chapter, neither the Laplace transform integral nor the complex inversion integral will be needed. Instead, use will be made of available tables [8–10] of Laplace transforms and inverse Laplace transforms so that given a function $s(t)$, its Laplace transform $\bar{s}(p)$, or given a function $\bar{s}(p)$, its inverse Laplace transform, can be found from the tables that are available in the literature.

1. Fundamental Properties

A few of the fundamental properties of the Laplace transformation that are of immediate use in solving ground-water boundary-value problems are:

$$L\{f_1 s_1 + f_2 s_2\} = f_1\bar{s}_1 + f_2\bar{s}_2 \tag{17}$$

in which f_1 and f_2 are constants or functions independent of the variable with respect to which the transformation is made;

$$L\{\partial s/\partial t\} = p\bar{s} - s(x,y,z,0) \tag{18}$$

$$L\{\nabla s\} = \nabla\bar{s}, \qquad L\{\nabla^2 s\} = \nabla^2\bar{s} \tag{19}$$

$$L\{s \exp(-at)\} = \bar{s}(p+a) \tag{20}$$

in which a is any constant and $L\{s\} = \bar{s}$;
when

$$F(t) = 0, \qquad 0 < t < t_0$$

and

$$F(t) = f(t - t_0), \qquad t > t_0$$

then

$$L\{F(t)\} = \exp(-t_0 p)\bar{f}(p) \tag{21}$$

in which, $\bar{f}(p) = L\{f(t)\}$; and

$$L^{-1}\{\bar{f}_1(p)\cdot\bar{f}_2(p)\} = \int_0^t f_1(t-\tau)f_2(\tau)\,d\tau \tag{22}$$

2. Short List of Transforms

A short list of Laplace transforms referred to in this chapter is given in the accompanying tabulation. See Section II, C for the definition of functions appearing in this list.[2]

	$f(p) = L\{\bar{f}(t)\}$	$f(t) = L^{-1}\{\bar{f}(p)\}$
1.	$1/p$	1
2.	$1/p^n$	$t^{n-1}/(n-1)\,!$
3.	$1/p(p+a)$	$(1/a)[1-\exp(-at)]$
4.	$1/\sqrt{p}$	$1/\sqrt{\pi t}$
5.	$1/p^k$	$t^{k-1}/\Gamma(k)$
6.	$(1/\sqrt{p})\exp(-k\sqrt{p})$	$(1/\sqrt{\pi t})\exp(-k^2/4t)$
7.	$(1/p)\exp(-k\sqrt{p})$	$\operatorname{erfc}(k/\sqrt{4t})$
8.	$(1/p^{1+n/2})\exp(-k\sqrt{p})$	$(4t)^{n/2}i^n\operatorname{erfc}(k/\sqrt{4t})$
9.	$K_0(k\sqrt{p})$	$(1/2t)\exp(-k^2/4t)$
10.	$(1/p)K_0(k\sqrt{p})$	$(1/2)W(k^2/4t)$
11.	$K_0(k\sqrt{p+a})$	$(1/2t)\exp(-at-k^2/4t)$
12.	$(1/p)K_0(k\sqrt{p+a})$	$(1/2)W(k^2/4t, k\sqrt{a})$
13.	$(1/p)K_0(k\sqrt{p+a\sqrt{p}})$	$(1/2)H(k^2/4t, ka/4)$
14.	$K_0(k\sqrt{p})/pK_0(k_1\sqrt{p})$	$A(t/k_1^2, k/k_1)$
15.	$K_0(k\sqrt{p})/p(k_1\sqrt{p})K_1(k_1\sqrt{p})$	$(1/2)S(t/k_1^2, k/k_1)$
16.	$K_0(k\sqrt{p+a})/pK_0(k_1\sqrt{p+a})$	$Z(t/k_1^2, k/k_1, k_1\sqrt{a})$

[2]The transforms 12 and 13 are obtained from Hantush [2, pp. 3722–3]; 14 and 15 are from Carslaw and Jaeger [10, p. 335 and p. 338]; and 11 is from Hantush [11, p. 1045].

B. Finite Fourier Transforms

These transforms are useful in solving boundary-value problems in which at least two of the boundaries are parallel and separated by a finite distance.

1. Finite Fourier Cosine Transform

Given a function $s(x,y,z,t)$, defined in the interval $0 < z < b$ and denoted henceforth by $s(z)$, its finite cosine transform $s_c(x,y,n,t)$, or simply $s_c(n)$, with respect to z is given by

$$f_c\{s(z)\} = s_c(n) = \int_0^b s(z) \cos(n\pi z/b)\, dz \tag{23}$$

in which $n = 0,1,2,3, \ldots$.

The inversion formula is

$$s(z) = (1/b)s_c(0) + (2/b)\sum_{n=1}^{\infty} s_c(n) \cos(n\pi z/b) \tag{24}$$

also

$$f_c\{\partial^2 s/\partial z^2\} = -(n\pi/b)^2 s_c(n) + (-)^n\, \partial s(b)/\partial z - \partial s(0)/\partial z \tag{25}$$

The finite cosine transform is useful for problems in which the velocities normal to two parallel boundaries are among the boundary conditions of the problem.

2. Finite Fourier Sine Transform

The finite sine transform $s_s(n,y,z,t)$, or simply $s_s(n)$, with respect to x of a function $s(x,y,z,t)$, or simply $s(x)$, when defined in the interval $0 < x < a$ is given by

$$f_s\{s(x)\} = s_s(n) = \int_0^a s(x) \sin(n\pi x/a)\, dx \tag{26}$$

and the inversion formula is

$$s(x) = (2/a)\sum_{n=1}^{\infty} s_s(n) \sin(n\pi x/a) \tag{27}$$

also

$$f_s\{\partial^2 s/\partial x^2\} = -(n\pi/a)^2 s_s(n) + (n\pi/a)[s(0) - (-)^n s(a)] \tag{28}$$

where $n = 1,2,3 \ldots$.

The finite sine transform is useful for problems involving boundary conditions of head distribution on two parallel boundaries.

Example 7. An effectively long and fairly straight stream cuts completely through a semi-infinite sand. If the water level in the stream is suddenly raised h_0 units above its initial elevation and maintained constant thereafter, find the rise of water levels in the sand and the rate of seepage into the sand caused by this change of stream level.

Let the stream be along the y-axis and the sand extend indefinitely to the right. If the water level in the stream slopes gently along the y-axis so that it may be assumed independent of y, and if the rise h_0 is small compared to the initial depth of saturation in the sand, this bank-storage flow problem may be represented by the following boundary-value problem:

$$\partial^2 h/\partial x^2 = \partial h/\nu\partial t \tag{a}$$

$$h(x,0) = 0 \tag{b}$$

$$h(0,t) = h_0 \tag{c}$$

$$h(\infty,t) = 0 \tag{d}$$

where h is the rise in the water level (caused by the change in stream level) at a section in the sand x distant from the stream.

Applying the Laplace transformation to the problem yields

$$L\{\partial^2 h/\partial x^2\} = L\{\partial h/\nu\partial t\}, \quad \text{or} \quad \partial^2 \bar{h}/\partial x^2 = (1/\nu)[p\bar{h} - h(x,0)],$$

$$\text{or} \quad \partial^2 \bar{h}/\partial x^2 - (p/\nu)\bar{h} = 0 \tag{a$'$}$$

$$L\{h(0,t)\} = L\{h_0\}, \quad \text{or} \quad \bar{h}(0,p) = h_0/p \tag{c$'$}$$

$$L\{h(\infty,t)\} = L\{0\}, \quad \text{or} \quad \bar{h}(\infty,p) = 0 \tag{d$'$}$$

Equation (a′) is a second-order linear differential equation, a solution of which satisfying Eqs. (c′) and (d′) is readily obtained as $\bar{h}=(h_0/p)\exp(-x\sqrt{p/\nu})$ whose inverse Laplace transform is

$$L^{-1}\{\bar{h}(x,p)\} = h_0 L^{-1}\{(1/p)\exp(-x\sqrt{p/\nu})\}$$

By definition, $L^{-1}\{\bar{h}(x,p)\} = h(x,t)$. From tables or from the list of transforms (Section II, A, 2), $L^{-1}\{(1/p)\exp(-x\sqrt{p/\nu}) = \operatorname{erfc}(x/\sqrt{4\nu t})$; consequently, the solution is $h(x,t) = h_0 \operatorname{erfc}(x/\sqrt{4\nu t})$ where erfc is the complementary error function.

From Darcy's law, the discharge per unit width of channel front across any section is $q(x,t) = -T\,\partial h/\partial x$, the Laplace transform of which is $\bar{q}(x,p) = -T\,\partial\bar{h}/\partial x = (Th_0/\sqrt{\nu})\,(1/\sqrt{p})\exp(-x\sqrt{p/\nu})$, from which the Laplace transform of the stream seepage into the sand per unit width of stream front is $\bar{q}(0,p) = (Th_0/\sqrt{\nu})(1/\sqrt{p})$.

The inverse Laplace transform, using tables, gives $q(0,t) = Th_0/\sqrt{\nu\pi t}$.

Example 8. Instead of being semi-infinite, the sand of Example 7 has a closed lateral boundary at $x = l$. Find the solutions corresponding to those of Example 7.

This bank-storage problem may be described by Eqs. (a) to (c) of Example 7 and by the condition

$$h(2l,t) = h_0 \tag{a$_1$}$$

Applying the Laplace transformation as in Example 7, the transformed problem will be given by Eqs. (a′) and (c′) of Example 7, and the Laplace transform of Eq. (a_1); namely,

$$\bar{h}(2l,p) = h_0/p \tag{a_2}$$

Applying the finite Fourier sine transform to Eq. (a′) of Example 7 yields

$$-(n\pi/2l)^2\bar{h}_s(n,p) + (n\pi/2l)[\bar{h}(0,p) - (-1)^n\bar{h}(2l,p)]$$
$$-(p/\nu)\bar{h}_s(n,p) = 0$$

Using Eq. (c′) of Example 7 and Eq. (a_2) and solving for $\bar{h}_s$ gives

$$\bar{h}_s(n,p) = (n\pi/2l)(1 - (-1)^n)h_0/p[p/\nu + (n\pi/2l)^2]$$

which, inverted by the finite Fourier sine transform inversion formula or Eq. (27), will be

$$\bar{h}(x,p) = \frac{2}{2l}\sum_{n=1}^{\infty}\frac{n\pi}{2l}[1 - (-1)^n]\frac{h_0[\sin(n\pi x/2l)]}{p[p/\nu + (n\pi/2l)^2]}$$

After obtaining the inverse Laplace transform from tables, the formal solution will be

$$h(x,t) = (2h_0/\pi)\sum_{n=1}^{\infty}(1/n)[1 - (-1)^n]$$
$$\cdot\{1 - \exp[-\nu t(n\pi/2l)^2]\}\sin(n\pi x/2l)$$

which, when n is replaced by $2m + 1$, becomes

$$h(x,t) = (4h_0/\pi)\sum_{m=0}^{\infty}[1/(2m + 1)]$$
$$\cdot\{1 - \exp(-\nu t[(2m + 1)\pi/2l]^2)\}\sin[(2m + 1)\pi x/2l]$$

in which the first sum may be recognized as equal to h_0. Consequently, the solution may be written as

$$h = h_0 - (4h_0/\pi)\sum_{m=0}^{\infty}[1/(2m + 1)]$$
$$\cdot\exp(-\nu t[(2m + 1)\pi/2l]^2)\sin[(2m + 1)\pi x/2l]$$

The rate of seepage from the stream per unit length is $q(t) = -T\,\partial h(0,t)/\partial x$; thus,

$$q(t) = (2Th_0/l)\sum_{m=0}^{\infty}\exp(-\nu t[(2m + 1)\pi/2l]^2)$$

Example 9. A well partially penetrates an infinite leaky artesian aquifer of uniform thickness. If the storage in the semipervious layer of the leaky system is neglected, find the drawdown distribution induced by pumping the well at a constant rate.

The flow toward the well may be described by the following boundary-value problem (Section IV, A):

$$\partial^2 s/\partial r^2 + (1/r)\,\partial s/\partial r + \partial^2 s/\partial z^2 - s/B^2 = (1/\nu)\,\partial s/\partial t \tag{a}$$

$$s(r,z,0) = 0 \tag{b}$$

$$s(\infty,z,t) = 0 \tag{c}$$

$$\partial s(r,0,t)/\partial z = \partial s(r,b,t)/\partial z = 0 \tag{d}$$

$$\begin{aligned}\lim_{r\to 0}\,[(l-d)r(\partial s/\partial r)] = A(t) &= 0, && 0<z<d\\ &= -Q/2\pi K, && d<z<l\\ &= 0, && l<z<b\end{aligned} \tag{e}$$

where the origin of the coordinate system is taken at the center of the well in the plane of the top of the aquifer, with the z-axis positive downward, $B^2 = T/(K'/b')$ where b', K' are the thickness and the conductivity of the semi-confining layer, respectively, b is the uniform thickness of the aquifer, l and d are, respectively, the total depth of penetration and the depth of penetration of the unscreened portion of the well, r and z are the radial distance and the vertical distance to any point in the aquifer (point locating the bottom of a piezometer), s is the drawdown induced at the point (r,z) at the end of the period t since the beginning of pumping, Q is the constant rate of pumping, and (K'/b') is the leakage coefficient (see Section III, C, 1, for definition).

Applying the Laplace transformation to Eqs. (a), (c), and (d) making use of Eq. (b), then applying the finite Fourier cosine transformation to the resulting expressions making use of the Laplace transform of Eq. (d), yields

$$\partial^2 \bar{s}_c/\partial r^2 + (1/r)\,\partial \bar{s}_c/\partial r - [p/\nu + 1/B^2 + (n\pi/b)^2]\bar{s}_c = 0, \tag{a'}$$

$$\bar{s}_c(\infty,n,p) = 0, \tag{b'}$$

$$\begin{aligned}&\lim_{r\to 0}\,[(l-d)r\,\partial \bar{s}_c/\partial r]\\ &= \int_0^b \bar{A}(p)\cos(n\pi z/b)\,dz = -\int_d^l (Q/2\pi K p)\cos(n\pi z/b)\,dz\\ &= -(Q/2\pi K p)(b/n\pi)[(\sin(n\pi l/b) - \sin(n\pi d/b)]\end{aligned} \tag{c'}$$

Equation (a′) is the zero order modified Bessel equation whose general solution [12] is $\bar{s}_c = c_1 K_0(Nr) + c_2 I_0(Nr)$ where $N^2 = p/\nu + 1/B^2 + (n\pi/b)^2$, and I_0 and K_0 are the zero order modified Bessel function of the first and second kinds, respectively.

Since $K_0(\infty) = 0$, $I_0(\infty) = \infty$, $\partial K_0(Nr)/\partial r = -NK_1(Nr)$, and since as $x \to 0$, $xK_1(x) \to 1$, then the values of c_1 and c_2 can be found by using Eqs. (b′) and (c′). Thus,

$$\bar{s}_c = [Q/2\pi K(l-d)] \cdot (b/n\pi p)[\sin(n\pi l/b) - \sin(n\pi d/b)]K_0(Nr) \qquad \text{(d}'\text{)}$$

whose inverse finite cosine transform, from Eq. (24), is

$$\bar{s} = (Q/2\pi Kb)\{(1/p)K_0(r\sqrt{p/\nu + 1/B^2}) + [2b/\pi(l-d)]$$

$$\cdot \sum_{n=1}^{\infty} R_n(1/p)K_0[r\sqrt{p/\nu + 1/B^2 + (n\pi/b)^2}]\}$$

where

$$R_n = (1/n)\ [\sin(n\pi l/b) - \sin(n\pi d/b)] \cdot \cos(n\pi z/b)$$

By using the list of Laplace transforms (Section II, A, 2), the inverse Laplace transform of the above equation is

$$s = (Q/4\pi Kb)\{W(u,r/B) + [2b/\pi(l-d)]\sum_{n=1}^{\infty} R_n$$

$$\cdot\ W[u, \sqrt{(r/B)^2 + (n\pi r/b)^2}]\}$$

where $u = r^2/4\nu t$ and $W(u, \beta)$ is an infinite integral, known as the *well function for leaky aquifers*, tabular values for which are available (Section II, C, 17).

Example 10. A well completely penetrating an infinite leaky artesian aquifer is pumped at a constant rate Q for a period of time t_0 after which the well is shut off. Find the drawdown around the well during the recovery period, if the storage in the semipervious layer of the leaky system is neglected.

This problem of drawdown and recovery of water levels around the well can be described by

$$\partial^2 s/\partial r^2 + (1/r)\ \partial s/\partial r - s/B^2 = (1/\nu)\ \partial s/\partial t \qquad \text{(a)}$$

$$s(r,0) = 0 \qquad \text{(b)}$$

$$s(\infty,t) = 0 \qquad \text{(c)}$$

$$\lim_{r\to 0} r\ \partial s/\partial r = -Q/2\pi T, \qquad 0 < t < t_0$$

$$= 0, \qquad t > t_0 \qquad \text{(d)}$$

Apply the Laplace transformation to obtain

$$\partial^2\bar{s}/\partial r^2 + (1/r)\ \partial\bar{s}/\partial r - (p/\nu + 1/B^2)\bar{s} = 0, \qquad \bar{s}(\infty, p) = 0$$

and

$$\lim_{r\to 0} r\ \partial\bar{s}/\partial r = -(Q/2\pi Tp)[1 - \exp(-pt_0)]$$

whose solution (see Eqs. (a′) to (c′) of Example 9 for obtaining the solution) is

$$\bar{s} = (Q/2\pi T)\{(1/p)K_0(r\sqrt{p/\nu + 1/B^2}) - (1/p)\exp(-pt_0) \cdot K_0(r\sqrt{p/\nu + 1/B^2})\}$$

whose inverse transform (from list of transforms) is

$$s = (Q/4\pi T)W(r^2/4\nu t, r/B) \qquad \text{for} \qquad 0 < t < t_0 \qquad \text{and}$$

$$s = (Q/4\pi T)\{W(r^2/4\nu t, r/B) - W(r^2/4\nu(t-t_0), r/B)\} \qquad \text{for} \qquad t > t_0$$

This result can be readily generalized for any problem of recovery of water levels after shutdown of wells that were pumped at a constant rate. Given the drawdown expression $s(r,t)$ for a steadily discharging well, then the residual drawdown s' during recovery after a continuous pumping period of t_0, will be given by

$$s' = s(r, t) - s(r, t-t_0)$$

C. Functions of Common Occurrence in Problems of Ground-Water Flow

Definitions and useful approximations for several functions that are of common occurrence in well problems and other ground-water-flow problems are given below in alphabetical order.

(1) $A(\tau,\rho)$ may be called the *flowing well function for nonleaky aquifers* and is defined [13] by

$$A(\tau,\rho) = 1 - \frac{2}{\pi}\int_0^\infty \frac{J_0(u)Y_0(\rho u) - Y_0(u)J_0(\rho u)}{J_0^2(u) + Y_0^2(u)} exp(-\tau u^2)\frac{du}{u}$$

The function is available in tabular form [13]. It is given in Table I. The function may be approximated for practical computation [14]:

a. for small values of $\tau, \tau < 0.05$,

$$A(\tau,\rho) \approx [1/\sqrt{\rho}]\{\text{erfc}[(\rho-1)/2\sqrt{\tau}] + [(\rho-1)\sqrt{\tau}/4\rho]\, i\, \text{erfc}[(\rho-1)/2\sqrt{\tau}]\}$$

b. for large values of τ, $\tau > 500$,

$$A(\tau, \rho) \approx [W(\rho^2/4\tau)]/\ln(2.25\tau)$$

(2) erf(x) and erfc(x) are, respectively, the error and complementary error functions and are defined by

$$\text{erf}(x) = 1 - \text{erfc}(x) = (2/\sqrt{\pi})\int_0^x \exp(-y^2)\, dy$$

TABLE I. Values of the Function $A(\tau, \rho)$

τ		ρ = 1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
	1	1.000	0.024	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	2	1.000	0.109	0.001								
	3	1.000	0.188	0.009	0.000							
	4	1.000	0.251	0.023	0.001							
10^{-3}	5	1.000	0.303	0.042	0.002							
	6	1.000	0.345	0.062	0.005	0.000						
	7	1.000	0.380	0.083	0.010	0.001						
	8	1.000	0.410	0.104	0.016	0.001						
	9	1.000	0.435	0.124	0.022	0.002						
	1	1.000	0.458	0.144	0.030	0.004	0.000	0.000				
	2	1.000	0.589	0.290	0.117	0.039	0.010	0.002	0.000	0.000		
	3	1.000	0.652	0.379	0.194	0.087	0.034	0.011	0.003	0.001	0.000	
	4	1.000	0.691	0.439	0.254	0.133	0.063	0.027	0.010	0.004	0.001	0.000
10^{-2}	5	1.000	0.718	0.483	0.302	0.175	0.093	0.046	0.021	0.009	0.003	0.001
	6	1.000	0.739	0.517	0.341	0.211	0.122	0.066	0.033	0.016	0.007	0.003
	7	1.000	0.754	0.544	0.373	0.242	0.149	0.087	0.047	0.024	0.012	0.005
	8	1.000	0.767	0.566	0.400	0.270	0.174	0.106	0.062	0.034	0.018	0.009
	9	1.000	0.778	0.585	0.423	0.294	0.196	0.125	0.077	0.045	0.025	0.013
	1	1.000	0.787	0.601	0.443	0.316	0.217	0.143	0.091	0.055	0.032	0.018
	2	1.000	0.837	0.691	0.562	0.450	0.355	0.275	0.209	0.156	0.114	0.082
	3	1.000	0.860	0.733	0.620	0.519	0.430	0.352	0.286	0.229	0.181	0.142
	4	1.000	0.873	0.758	0.655	0.562	0.479	0.405	0.339	0.282	0.233	0.191
10^{-1}	5	1.000	0.883	0.776	0.680	0.592	0.514	0.443	0.380	0.323	0.274	0.230
	6	1.000	0.890	0.789	0.698	0.615	0.540	0.472	0.411	0.356	0.307	0.263
	7	1.000	0.895	0.800	0.713	0.634	0.562	0.496	0.436	0.382	0.334	0.290
	8	1.000	0.899	0.808	0.725	0.649	0.579	0.515	0.457	0.405	0.357	0.313
	9	1.000	0.903	0.815	0.735	0.661	0.594	0.532	0.475	0.424	0.377	0.334
	1	1.000	0.906	0.821	0.743	0.672	0.606	0.546	0.491	0.440	0.394	0.351
	2	1.000	0.924	0.854	0.790	0.732	0.677	0.627	0.580	0.536	0.496	0.458
	3	1.000	0.932	0.870	0.812	0.760	0.711	0.665	0.623	0.583	0.546	0.511
	4	1.000	0.937	0.879	0.826	0.777	0.731	0.689	0.649	0.612	0.577	0.544
1	5	1.000	0.940	0.886	0.835	0.789	0.746	0.706	0.668	0.633	0.599	0.568
	6	1.000	0.943	0.890	0.842	0.798	0.757	0.718	0.682	0.648	0.616	0.586
	7	1.000	0.945	0.894	0.848	0.805	0.765	0.728	0.693	0.661	0.630	0.601
	8	1.000	0.946	0.898	0.853	0.811	0.773	0.737	0.703	0.671	0.641	0.613
	9	1.000	0.948	0.900	0.857	0.816	0.779	0.743	0.711	0.680	0.650	0.623
	1	1.000	0.949	0.903	0.860	0.820	0.784	0.749	0.717	0.687	0.658	0.631
	2	1.000	0.956	0.916	0.879	0.845	0.813	0.783	0.756	0.729	0.704	0.681
	3	1.000	0.959	0.922	0.888	0.856	0.827	0.800	0.774	0.750	0.726	0.705
	4	1.000	0.962	0.926	0.894	0.864	0.836	0.810	0.786	0.762	0.741	0.720
10	5	1.000	0.963	0.929	0.898	0.869	0.843	0.818	0.794	0.772	0.751	0.731
	6	1.000	0.964	0.931	0.901	0.874	0.848	0.823	0.800	0.779	0.759	0.739
	7	1.000	0.965	0.933	0.904	0.877	0.851	0.828	0.806	0.785	0.765	0.746
	8	1.000	0.966	0.935	0.906	0.879	0.855	0.832	0.810	0.789	0.770	0.752
	9	1.000	0.966	0.936	0.908	0.882	0.857	0.835	0.813	0.793	0.774	0.756
	1	1.000	0.967	0.937	0.909	0.884	0.860	0.838	0.817	0.797	0.778	0.760
	2	1.000	0.970	0.943	0.918	0.895	0.874	0.854	0.835	0.817	0.800	0.784
	3	1.000	0.972	0.946	0.923	0.904	0.881	0.862	0.844	0.827	0.811	0.796
	4	1.000	0.973	0.948	0.926	0.905	0.886	0.867	0.850	0.834	0.819	0.804
10^2	5	1.000	0.974	0.950	0.928	0.908	0.889	0.871	0.855	0.839	0.824	0.810
	6	1.000	0.974	0.951	0.930	0.910	0.891	0.874	0.858	0.842	0.828	0.814
	7	1.000	0.975	0.952	0.931	0.912	0.893	0.877	0.861	0.846	0.831	0.818
	8	1.000	0.975	0.953	0.932	0.913	0.895	0.879	0.863	0.848	0.834	0.821
	9	1.000	0.976	0.954	0.933	0.914	0.897	0.880	0.865	0.850	0.837	0.824
	1	1.000	0.976	0.954	0.934	0.915	0.898	0.882	0.867	0.852	0.839	0.826

τ		ρ = 3.0	4.0	5.0	6.0	7.0	8.0	9.0	10
	1	0.000	0.000						
	2								
	3								
	4								
10^{-3}	5								
	6								
	7								
	8								
	9								
	1								
	2								
	3								
	4								
10^{-2}	5								
	6								
	7								
	8								
	9								
	1	0.000							
	2	0.001							
	3	0.006							
	4	0.015	0.000						
10^{-1}	5	0.027	0.001						
	6	0.040	0.003						
	7	0.054	0.006	0.000	0.000	0.000	0.000	0.000	0.000
	8	0.068	0.009	0.001					
	9	0.082	0.013	0.001					
	1	0.095	0.018	0.002	0.000	0.000			
	2	0.194	0.071	0.022	0.005	0.001	0.000		
	3	0.256	0.119	0.049	0.018	0.006	0.002	0.000	
	4	0.300	0.157	0.076	0.034	0.014	0.005	0.002	0.000
1	5	0.332	0.188	0.101	0.051	0.024	0.010	0.004	0.002
	6	0.357	0.214	0.123	0.067	0.035	0.017	0.008	0.003
	7	0.378	0.235	0.142	0.082	0.046	0.024	0.012	0.006
	8	0.395	0.254	0.159	0.097	0.056	0.032	0.017	0.009
	9	0.409	0.270	0.175	0.110	0.067	0.039	0.022	0.012
	1	0.422	0.284	0.188	0.122	0.077	0.047	0.028	0.016
	2	0.497	0.370	0.277	0.207	0.153	0.112	0.081	0.057
	3	0.534	0.415	0.325	0.256	0.201	0.157	0.122	0.094
	4	0.558	0.444	0.358	0.290	0.235	0.190	0.153	0.123
10	5	0.574	0.464	0.381	0.314	0.260	0.215	0.177	0.146
	6	0.588	0.481	0.399	0.334	0.281	0.236	0.199	0.167
	7	0.598	0.494	0.414	0.350	0.297	0.253	0.216	0.184
	8	0.607	0.505	0.427	0.364	0.312	0.268	0.230	0.198
	9	0.614	0.514	0.437	0.374	0.323	0.280	0.242	0.210
	1	0.621	0.522	0.446	0.385	0.334	0.291	0.254	0.222
	2	0.658	0.569	0.500	0.444	0.397	0.357	0.322	0.291
	3	0.677	0.593	0.528	0.475	0.430	0.392	0.358	0.328
	4	0.690	0.609	0.546	0.495	0.451	0.414	0.382	0.353
10^2	5	0.699	0.620	0.559	0.510	0.468	0.432	0.400	0.372
	6	0.706	0.629	0.569	0.520	0.479	0.444	0.413	0.385
	7	0.712	0.636	0.578	0.530	0.490	0.455	0.424	0.397
	8	9.716	0.642	0.585	0.538	0.498	0.464	0.434	0.407
	9	0.720	0.647	0.591	0.544	0.505	0.471	0.442	0.415
	1	0.724	0.652	0.596	0.550	0.511	0.478	0.449	0.422

τ	ρ = 10	20	30	40	50	60	70	80	90	100
10	0.016	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
20	0.057	0.001								
30	0.094	0.004								
40	0.123	0.009	0.000							
50	0.146	0.016	0.001							
60	0.167	0.023	0.002							
70	0.184	0.031	0.003							
80	0.198	0.038	0.005	0.000						
90	0.210	0.046	0.007	0.001						
100	0.222	0.053	0.010	0.001	0.000	0.000				
200	0.291	0.110	0.038	0.011	0.001	0.001	0.000			
300	0.328	0.146	0.064	0.026	0.009	0.003	0.001	0.000		
400	0.353	0.173	0.086	0.040	0.018	0.007	0.003	0.001	0.000	
500	0.372	0.194	0.104	0.054	0.026	0.012	0.005	0.002	0.001	0.000
600	0.385	0.210	0.119	0.066	0.035	0.018	0.008	0.004	0.002	0.001
700	0.397	0.223	0.132	0.077	0.044	0.024	0.012	0.006	0.003	0.001
800	0.407	0.235	0.143	0.087	0.052	0.030	0.016	0.009	0.004	0.002
900	0.415	0.245	0.153	0.096	0.059	0.035	0.020	0.011	0.006	0.003
1000	0.422	0.254	0.162	0.104	0.066	0.041	0.024	0.014	0.008	0.004

TABLE II. Values of the Function $G(\tau,\beta)$

τ \ β	0	1x10^{-5}	2x10^{-5}	4x10^{-5}	6x10^{-5}	8x10^{-5}	10^{-4}	2x10^{-4}	4x10^{-4}	6x10^{-4}	8x10^{-4}	10^{-3}	2x10^{-3}	4x10^{-3}	6x10^{-3}	8x10^{-3}	10^{-2}
1x10^{2}	0.346																0.346
2	0.311												0.311	0.311	0.311	0.312	0.312
3	0.294												0.294	0.294	0.294	0.295	0.295
4	0.283												0.283	0.283	0.283	0.284	0.285
5	0.274												0.274	0.274	0.275	0.275	0.276
6	0.268												0.268	0.268	0.268	0.269	0.271
7	0.263												0.263	0.263	0.263	0.264	0.266
8	0.258												0.258	0.258	0.259	0.260	0.261
9	0.254												0.254	0.255	0.256	0.257	0.258
1x10^{3}	0.251												0.251	0.252	0.252	0.254	0.255
2	0.232												0.232	0.233	0.234	0.236	0.239
3	0.222												0.222	0.223	0.225	0.227	0.231
4	0.215												0.215	0.216	0.219	0.222	0.226
5	0.210												0.210	0.212	0.215	0.218	0.222
6	0.206												0.206	0.208	0.211	0.215	0.220
7	0.203												0.203	0.205	0.209	0.213	0.219
8	0.201												0.201	0.203	0.207	0.212	0.218
9	0.198												0.198	0.201	0.205	0.210	0.217
1x10^{4}	0.196											0.196	0.197	0.200	0.204	0.209	0.216
2	0.185											0.185	0.185	0.190	0.197	0.205	0.213
3	0.178											0.178	0.179	0.186	0.194	0.203	0.212
4	0.173											0.173	0.176	0.183	0.193	0.202	
5	0.170											0.170	0.173	0.181	0.192		
6	0.168											0.168	0.171	0.180	0.192		
7	0.166										0.166	0.167	0.170	0.179	0.191		
8	0.164										0.164	0.165	0.169	0.179			
9	0.163										0.163	0.164	0.168	0.179			
1x10^{5}	0.161								0.161	0.162	0.162	0.162	0.167	0.178			
2	0.152							0.152	0.153	0.153	0.154	0.155	0.163	0.177			
3	0.148							0.148	0.148	0.149	0.150	0.152	0.162				
4	0.145							0.145	0.145	0.146	0.147	0.150	0.162				
5	0.143							0.143	0.143	0.144	0.145	0.148	0.161				
6	0.141							0.141	0.142	0.143	0.144	0.147	0.160				
7	0.140							0.140	0.140	0.141	0.143	0.146	0.160				
8	0.138							0.138	0.139	0.141	0.143	0.145	0.160				
9	0.137							0.137	0.138	0.140	0.142	0.144	0.160				
1x10^{6}	0.136						0.136	0.137	0.138	0.139	0.141	0.144	0.159				
2	0.130						0.130	0.131	0.133	0.135	0.139	0.143	0.159				
3	0.127						0.127	0.127	0.130	0.134	0.138	0.142	0.158				
4	0.124						0.124	0.125	0.129	0.134							
5	0.123						0.123	0.124	0.128	0.133							
6	0.121						0.121	0.123	0.128								
7	0.120						0.120	0.122	0.127								
8	0.119						0.119	0.121	0.127								
9	0.118						0.118	0.121	0.127								
1x10^{7}	0.118						0.118	0.120	0.127								
2	0.114						0.114	0.116	0.126								
3	0.111					0.111	0.112										
4	0.109				0.109	0.110	0.111										
5	0.108				0.108	0.109	0.110										
6	0.107			0.107	0.108	0.109	0.110										
7	0.106			0.106	0.107	0.108	0.109										
8	0.105			0.105	0.106	0.108	0.109										
9	0.104		0.104	0.105	0.106	0.107	0.108										
1x10^{8}	0.104		0.104	0.104	0.105	0.106	0.108										
2	0.100	0.100	0.101	0.102	0.103	0.105	0.107										
3	0.0982	0.0982	0.0986	0.100	0.103												
4	0.0968	0.0968	0.0974	0.0994	0.102												
5	0.0958	0.0958	0.0966	0.0989													
6	0.0950	0.0951	0.0959	0.0986													
7	0.0943	0.0944	0.0954	0.0984													
8	0.0937	0.0939	0.0949	0.0982													
9	0.0932	0.0934	0.0946	0.0981													
1x10^{9}	0.0927	0.0930	0.0943	0.0980													
2	0.0899	0.0906	0.0927	0.0977													
3	0.0883	0.0893	0.0920	0.0976													
4	0.0872	0.0885	0.0917														
5	0.0864	0.0880	0.0916														
6	0.0857	0.0876	0.0915														
7	0.0851	0.0873	0.0915														
8	0.0846	0.0870	0.0915														
9	0.0842	0.0869	0.0914														
1x10^{10}	0.0838	0.0867	0.0914														
2	0.0814	0.0862															
3	0.0861	0.0860															
4	0.0792																
5	0.0785																
6	0.0779																
7	0.0774																
8	0.0770																
9	0.0767																
10	0.0764	0.0860	0.0914	0.0976	0.102	0.105	0.107	0.116	0.126	0.133	0.138	0.142	0.158	0.177	0.191	0.202	0.212

They are available in tabular form [10, 15]. The following relations are useful:

a. $\operatorname{erf}(-x) = -\operatorname{erf}(x), \quad \operatorname{erf}(0) = 0, \quad \operatorname{erf}(\infty) = 1,$
$\operatorname{erfc}(-x) = 1 + \operatorname{erf}(x), \quad \operatorname{erfc}(-\infty) = 2$

b. for $x < 0.1, \quad \operatorname{erf}(x) \approx 2x/\sqrt{\pi},$
for $x > 9, \quad \exp(x)\operatorname{erfc}(\sqrt{x}) \approx 1/\sqrt{\pi x}$

(3) $G(\tau)$ may be called the *flowing well discharge function for nonleaky aquifers* and is defined by $G(\tau) = G(\tau, 0)$.

The function is extensively tabulated [16]. Sufficient values of this function are given in Table II. It may be approximated by

a. for $\tau < 0.05, \quad G(\tau) \approx 0.5 + 1/\sqrt{\pi\tau}$

b. for $\tau > 500, \quad G(\tau) \approx 2/\ln(2.25\tau)$

(4) $G(\tau,\beta)$ may be called the *flowing well discharge function for leaky aquifers* and is defined [11] by

$$G(\tau,\beta) = \beta K_1(\beta)/K_0(\beta) + (4/\pi^2)\exp(-\tau\beta^2)\int_0^\infty \exp(-\tau u^2) F\,du$$

where

$$F = u/[u^2 + \beta^2][J_0^2(u) + Y_0^2(u)]$$

Sufficient values of the function are given in Table II. It may be approximated by

a. for $\tau < 0.01, \quad G(\tau,\beta) \approx G(\tau, 0) = G(\tau)$

b. for $\tau\beta^2 > 1, \quad G(\tau,\beta) \approx 2/W(1/4\tau, \beta)$

(5) $H(u,\beta)$ defined [2] by

$$H(u,\beta) = \int_u^\infty (1/y)\exp(-y)\operatorname{erfc}[\beta\sqrt{u}/\sqrt{y(y-u)}]\,dy$$

and is extensively tabulated [17]. Sufficient values of the function are given in Table III. It may be approximated [2] by

a. $u > 10^4\beta^2,$

$$H(u,\beta) \approx W(u) - (4\beta/\sqrt{\pi u})[0.258 + 0.693\exp(-0.5u)]$$

b. $u <$ than both $10^{-5}/\beta^2$ and $10^{-4}\beta^2$

$$H(u,\beta) = (1/2)\ln(0.044/\beta^2 u)$$

TABLE III. Values of the Function $H(\mu, \beta)$

β / u	(-3)			(-2)			(-1)			(0)			(1)			(2)		
	1	2	5	1	2	5	1	2	5	1	2	5	1	2	5	1	2	5
1 (-6)	11.9842	11.4237	10.5908	9.9259	9.2469	8.3395	7.6497	6.9590	6.0463	5.3575	4.6721	3.7756	3.1110	2.4671	1.6710	1.1361	0.6879	0.2698
2 (-6)	11.5355	11.0211	10.2210	9.5677	8.8946	7.9908	7.3024	6.6126	5.7012	5.0141	4.3312	3.4412	2.7857	2.1568	1.3944	0.8995	0.5045	0.1685
3 (-6)	11.2593	10.7764	10.0006	9.3561	8.6875	7.7864	7.0991	6.4100	5.4996	4.8136	4.1327	3.2474	2.5984	1.9801	1.2409	0.7725	0.4113	0.1230
4 (-6)	11.0568	10.5980	9.8419	9.20[illegible]	8.5399	7.6412	6.9547	6.2663	5.3567	4.6716	3.9922	3.1109	2.4671	1.8571	1.1361	0.6879	0.3516	963(-4)
5 (-6)	10.8958	10.4566	9.7174	9.086[illegible]	8.4251	7.5284	6.8427	6.1548	5.2459	4.5617	3.8836	3.0055	2.3661	1.7633	1.0574	0.6256	0.3091	787(-4)
6 (-6)	10.7618	10.3389	9.6145	8.9894	8.3310	7.4362	6.7512	6.0637	5.1555	4.4719	3.7951	2.9199	2.2844	1.6877	0.9948	0.5770	0.2768	662(-4)
7 (-6)	10.6466	10.2379	9.5267	8.9069	8.2512	7.3581	6.6737	5.9867	5.0790	4.3962	3.7204	2.8478	2.2158	1.6246	0.9432	0.5375	0.2511	568(-4)
8 (-6)	10.5455	10.1493	9.4500	8.8350	8.1819	7.2904	6.6066	5.9200	5.0129	4.3306	3.6558	2.7856	2.1568	1.5706	0.8995	0.5045	0.2302	494(-4)
9 (-6)	10.4553	10.0702	9.3818	8.7714	8.1206	7.2306	6.5474	5.8611	4.9545	4.2728	3.5989	2.7309	2.1050	1.5234	0.8617	0.4763	0.2127	436(-4)
1 (-5)	10.3739	9.9987	9.3203	8.7142	8.0657	7.1771	6.4944	5.8085	4.9024	4.2212	3.5481	2.6822	2.0590	1.4816	0.8285	0.4519	0.1978	388(-4)
2 (-5)	9.8197	9.5097	8.9045	8.3315	7.7010	6.8238	6.1453	5.4623	4.5598	3.8827	3.2162	2.3660	1.7632	1.2170	0.6256	0.3091	0.1166	166(-4)
3 (-5)	9.4815	9.2084	8.6504	8.1013	7.4844	6.6159	5.9406	5.2597	4.3600	3.6858	3.0241	2.1850	1.5965	1.0716	0.5203	0.2402	818(-4)	929(-5)
4 (-5)	9.2358	8.9878	8.4646	7.9346	7.3290	6.4677	5.7951	5.1160	4.2185	3.5468	2.8889	2.0588	1.4815	0.9730	0.4518	0.1978	621(-4)	590(-5)
5 (-5)	9.0422	8.8128	8.3171	7.8031	7.2072	6.3523	5.6821	5.0045	4.1090	3.4394	2.7848	1.9622	1.3943	0.8994	0.4024	0.1685	494(-4)	405(-5)
6 (-5)	8.8822	8.6674	8.1943	7.6941	7.1068	6.2576	5.5896	4.9134	4.0196	3.3519	2.7002	1.8841	1.3244	0.8412	0.3645	0.1468	406(-4)	293(-5)
7 (-5)	8.7457	8.5428	8.0888	7.6007	7.0212	6.1773	5.5113	4.8364	3.9442	3.2781	2.6290	1.8189	1.2664	0.7934	0.3341	0.1300	342(-4)	220(-5)
8 (-5)	8.6266	8.4336	7.9962	7.5190	6.9466	6.1076	5.4434	4.7697	3.8789	3.2143	2.5677	1.7629	1.2169	0.7530	0.3090	0.1166	292(-4)	169(-5)
9 (-5)	8.5209	8.3364	7.9134	7.4461	6.8804	6.0459	5.3834	4.7108	3.8214	3.1583	2.5138	1.7139	1.1739	0.7182	0.2879	0.1056	253(-4)	133(-5)
1 (-4)	8.4258	8.2487	7.8386	7.3803	6.8208	5.9906	5.3297	4.6581	3.7700	3.1082	2.4658	1.6704	1.1359	0.6878	0.2698	963(-4)	222(-4)	107(-5)
2 (-4)	7.7901	7.6563	7.3276	6.9321	6.4190	5.6226	4.9747	4.3115	3.4334	2.7819	2.1549	1.3937	0.8992	0.5044	0.1684	494(-4)	842(-5)	206(-6)
3 (-4)	7.4110	7.2985	7.0135	6.6563	6.1750	5.4035	4.7655	4.1086	3.2379	2.5937	1.9778	1.2401	0.7721	0.4111	0.1229	315(-4)	434(-5)	674(-7)
4 (-4)	7.1393	7.0401	6.7840	6.4541	5.9971	5.2459	4.6161	3.9645	3.0999	2.4617	1.8545	1.1352	0.6875	0.3514	963(-4)	222(-4)	260(-5)	275(-7)
5 (-4)	6.9273	6.8375	6.6024	6.2934	5.8561	5.1223	4.4996	3.8527	2.9933	2.3601	1.7604	1.0564	0.6252	0.3089	787(-4)	166(-4)	169(-5)	129(-7)
6 (-4)	6.7533	6.6705	6.4517	6.1596	5.7389	5.0203	4.4040	3.7612	2.9065	2.2778	1.6846	0.9937	0.5765	0.2766	661(-4)	129(-4)	117(-5)	677(-8)
7 (-4)	6.6056	6.5284	6.3227	6.0447	5.6383	4.9333	4.3228	3.6838	2.8334	2.2087	1.6212	0.9420	0.5370	0.2510	567(-4)	103(-4)	834(-6)	383(-8)
8 (-4)	6.4774	6.4048	6.2100	5.9439	5.5501	4.8573	4.2523	3.6167	2.7702	2.1492	1.5670	0.8982	0.5040	0.2300	494(-4)	841(-5)	616(-6)	229(-8)
9 (-4)	6.3640	6.2952	6.1097	5.8539	5.4713	4.7898	4.1898	3.5575	2.7146	2.0971	1.5196	0.8603	0.4758	0.2125	436(-4)	699(-5)	466(-6)	142(-8)
1 (-3)	6.2624	6.1969	6.0193	5.7727	5.4001	4.7290	4.1337	3.5045	2.6650	2.0506	1.4776	0.8271	0.4513	0.1976	388(-4)	590(-5)	361(-6)	
2 (-3)	5.5903	5.5431	5.4115	5.2203	4.9139	4.3184	3.7598	3.1549	2.3419	1.7516	1.2116	0.6238	0.3084	0.1164	166(-4)	169(-5)	555(-7)	
3 (-3)	5.1948	5.1560	5.0465	4.8837	4.6149	4.0683	3.5363	2.9494	2.1559	1.5825	1.0652	0.5182	0.2394	816(-4)	927(-5)	713(-6)	149(-7)	
4 (-3)	4.9135	4.8798	4.7839	4.6396	4.3962	3.8859	3.3750	2.8030	2.0253	1.4656	0.9658	0.4496	0.1970	619(-4)	588(-5)	360(-6)	534(-8)	
5 (-3)	4.6951	4.6649	4.5786	4.4474	4.2231	3.7415	3.2483	2.6891	1.9250	1.3767	0.8915	0.4001	0.1677	493(-4)	403(-5)	205(-6)	228(-8)	
6 (-3)	4.5165	4.4890	4.4099	4.2888	4.0794	3.6214	3.1436	2.5957	1.8437	1.3054	0.8327	0.3620	0.1460	404(-4)	292(-5)	126(-6)	109(-8)	
7 (-3)	4.3655	4.3400	4.2666	4.1536	3.9564	3.5185	3.0542	2.5165	1.7754	1.2460	0.7843	0.3315	0.1292	340(-4)	219(-5)	821(-7)		
8 (-3)	4.2347	4.2109	4.1421	4.0357	3.8488	3.4282	2.9762	2.4478	1.7166	1.1953	0.7435	0.3064	0.1158	290(-4)	169(-5)	553(-7)		
9 (-3)	4.1194	4.0969	4.0320	3.9313	3.7531	3.3478	2.9068	2.3870	1.6651	1.1512	0.7083	0.2852	0.1047	252(-4)	133(-5)	384(-7)		
1 (-2)	4.0163	3.9950	3.9334	3.8374	3.6669	3.2752	2.8443	2.3325	1.6193	1.1122	0.6775	0.2670	955(-4)	221(-4)	106(-5)	274(-7)		
2 (-2)	3.3397	3.3250	3.2818	3.2133	3.0880	2.7829	2.4227	1.9714	1.3239	0.8677	0.4914	0.1653	487(-4)	831(-5)	203(-6)	226(-8)		
3 (-2)	2.9472	2.9354	2.9007	2.8452	2.7423	2.4844	2.1680	1.7579	1.1570	0.7353	0.3965	0.1197	308(-4)	426(-5)	662(-7)			
4 (-2)	2.6711	2.6611	2.6316	2.5842	2.4955	2.2691	1.9841	1.6056	1.0416	0.6467	0.3357	931(-4)	216(-4)	253(-5)	269(-7)			
5 (-2)	2.4590	2.4502	2.4243	2.3826	2.3040	2.1007	1.8401	1.4872	0.9540	0.5812	0.2923	755(-4)	160(-4)	164(-5)	126(-7)			
6 (-2)	2.2874	2.2795	2.2563	2.2188	2.1478	1.9626	1.7217	1.3905	0.8838	0.5298	0.2593	630(-4)	124(-4)	112(-5)	655(-8)			
7 (-2)	2.1436	2.1365	2.1154	2.0812	2.0164	1.8458	1.6213	1.3088	0.8255	0.4880	0.2332	536(-4)	982(-5)	799(-6)	368(-8)			
8 (-2)	2.0203	2.0138	1.9944	1.9630	1.9031	1.7448	1.5343	1.2381	0.7758	0.4530	0.2119	464(-4)	797(-5)	587(-6)	219(-8)			
9 (-2)	1.9126	1.9066	1.8886	1.8595	1.8039	1.6559	1.4577	1.1760	0.7327	0.4230	0.1941	406(-4)	658(-5)	442(-6)	135(-8)			
1 (-1)	1.8172	1.8116	1.7949	1.7677	1.7157	1.5768	1.3893	1.1207	0.6947	0.3970	0.1789	359(-4)	552(-5)	340(-6)				
2 (-1)	1.2193	1.2159	1.2059	1.1895	1.1579	1.0714	0.9497	0.7665	0.4603	0.2452	971(-4)	143(-4)	149(-5)	493(-7)				
3 (-1)	0.9033	0.9010	0.8940	0.8825	0.8603	0.7986	0.7103	0.5739	0.3390	0.1729	629(-4)	752(-5)	592(-6)	126(-7)				
4 (-1)	0.7006	0.6989	0.6936	0.6850	0.6683	0.6218	0.5543	0.4482	0.2619	0.1296	441(-4)	448(-5)	283(-6)	424(-8)				
5 (-1)	0.5584	0.5570	0.5530	0.5463	0.5333	0.4969	0.4436	0.3591	0.2083	0.1006	325(-4)	288(-5)	151(-6)	171(-8)				
6 (-1)	0.4533	0.4522	0.4490	0.4437	0.4333	0.4041	0.3613	0.2927	0.1688	799(-4)	247(-4)	195(-5)	873(-7)					
7 (-1)	0.3729	0.3720	0.3694	0.3651	0.3567	0.3330	0.2980	0.2415	0.1386	646(-4)	192(-4)	137(-5)	534(-7)					
8 (-1)	0.3099	0.3092	0.3070	0.3035	0.2966	0.2770	0.2481	0.2012	0.1151	529(-4)	152(-4)	986(-6)	340(-7)					
9 (-1)	0.2596	0.2590	0.2572	0.2543	0.2485	0.2323	0.2082	0.1690	963(-4)	437(-4)	122(-4)	728(-6)	223(-7)					
1 (0)	0.2189	0.2184	0.2169	0.2144	0.2097	0.1961	0.1758	0.1427	812(-4)	365(-4)	993(-5)	547(-6)	151(-7)					
2 (0)	488(-4)	487(-4)	484(-4)	479(-4)	468(-4)	439(-4)	395(-4)	322(-4)	180(-4)	760(-5)	173(-5)	551(-7)						
3 (0)	130(-4)	130(-4)	129(-4)	128(-4)	125(-4)	117(-4)	106(-4)	862(-5)	481(-5)	196(-5)	406(-6)	911(-8)						
4 (0)	377(-5)	376(-5)	374(-5)	370(-5)	362(-5)	340(-5)	307(-5)	250(-5)	139(-5)	558(-6)	108(-6)	189(-8)						
5 (0)	115(-5)	114(-5)	114(-5)	112(-5)	110(-5)	104(-5)	934(-6)	763(-6)	423(-6)	167(-6)	309(-7)							
6 (0)	359(-6)	359(-6)	356(-6)	353(-6)	345(-6)	325(-6)	293(-6)	239(-6)	133(-6)	519(-7)	926(-8)							
7 (0)	115(-6)	115(-6)	114(-6)	113(-6)	111(-6)	104(-6)	941(-7)	769(-7)	425(-7)	165(-7)	286(-8)							
8 (0)	376(-7)	375(-7)	373(-7)	369(-7)	362(-7)	340(-7)	307(-7)	251(-7)	139(-7)	536(-8)								
9 (0)	124(-7)	124(-7)	123(-7)	122(-7)	119(-7)	112(-7)	102(-7)	830(-8)	458(-8)	176(-8)								
10 (0)	415(-8)	414(-8)	411(-8)	407(-8)	399(-8)	375(-8)	339(-8)	277(-8)	153(-8)									

The numbers in parenthesis are powers of ten by which the other numbers are multiplied, e.g., 488(-4) = 0.0488

(6) $I_0(z)$ and $I_1(z)$ are, respectively, the zero- and first-order modified Bessel functions of the first kind [12]. They are available in tabular form [15] and may be approximated by:

a. for $z < 0.1$, $I_0(z) \approx 1$, $I_1(z) \approx 0.5z$

b. for $z > 5$, $I_0(z) \approx (1 + 1/8z)\exp(z)/\sqrt{2\pi z}$,

$I_1(z) \approx (1 - 3/8z)\exp(z)/\sqrt{2\pi z}$

c. $I_0'(z) = I_1(z)$

(7) $i^n \operatorname{erfc}(x)$, defined [10] by

$$i^n \operatorname{erfc}(x) = \int_x^\infty i^{n-1} \operatorname{erfc}(\beta)\, d\beta, \qquad n = 1, 2, 3$$

with

$$i^0 \operatorname{erfc}(x) = \operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$$

is the nth repeated integral of the error function, tabular values of which are available [10, 18]. A general recurrence formula for the function is

$$2ni^n\operatorname{erfc}(x) = i^{n-2} \operatorname{erfc}(x) - 2xi^{n-1} \operatorname{erfc}(x)$$

and

$$i^n\operatorname{erfc}(0) = 1/[2^n\Gamma(1 + n/2)]$$

(8) $J_0(z)$ and $J_1(z)$ are, respectively, the zero- and first-order Bessel functions of the first kind [12]. They are available in tabular form [15] and may be approximated by

a. for $z < 0.1$, $\quad J_0(z) \approx 1, \quad J_1(z) \approx 0.5z$

b. for $z > 16$, $\quad J_0(z) \approx \sqrt{2/\pi z} \cos(z - \pi/4),$
$\quad J_1(z) \approx \sqrt{2/\pi z} \sin(z - \pi/4)$

c. $J_0'(z) = -J_1(z)$

(9) $K_0(z)$ and $K_1(z)$ are, respectively, the zero- and first-order modified Bessel functions of the second kind [12]. They are available in tabular form [15] and may be approximated by

a. for $z < 0.05$, $\quad K_0(z) \approx -[0.5772 + \ln(z/2)] = \ln(1.12/z),$
$\quad K_1(z) \approx 1/z$

b. for $z > 5$, $\quad K_0(z) \approx \sqrt{\pi/2z}(1 - 1/8z) \exp(-z),$
$\quad K_1(z) \approx \sqrt{\pi/2z}(1 + 3/8z) \exp(-z)$

c. $K_0'(z) = -K_1(z)$

(10) $L(u, \pm w)$ is defined [19] by

$$L(u, \pm w) = -L(-u, \pm w) = \int_0^u K_0(\sqrt{w^2 + y^2})\, dy$$

and can be easily tabulated for a practical range of u and w. A special case is

$$L(u,0) = -L(-u,0) = \int_0^u K_0(y)\, dy$$

tabular values of which [20] are given in Table V.

TABLE IV - Values of the Function $M(\mu, \beta)$

u \ β		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.2	1.4	1.6	1.8	2.0
	0	0.1997	0.3974	0.5913	0.7801	0.9624	1.1376	1.3053	1.4653	1.6177	1.7627	2.0319	2.2759	2.4979	2.7009	2.8872
	1	0.1994	0.3969	0.5907	0.7792	0.9613	1.1363	1.3037	1.4635	1.6157	1.7605	2.0292	2.2728	2.4943	2.6968	2.8827
	2	0.1993	0.3967	0.5904	0.7788	0.9608	1.1357	1.3031	1.4628	1.6148	1.7595	2.0281	2.2715	2.4929	2.6951	2.8809
	3	0.1993	0.3966	0.5902	0.7785	0.9605	1.1353	1.3026	1.4622	1.6142	1.7588	2.0272	2.2705	2.4917	2.6938	2.8794
	4	0.1992	0.3965	0.5900	0.7783	0.9602	1.1349	1.3022	1.4617	1.6137	1.7582	2.0265	2.2696	2.4907	2.6927	2.8782
10^{-6}	5	0.1992	0.3964	0.5898	0.7780	0.9599	1.1346	1.3018	1.4613	1.6132	1.7577	2.0259	2.2689	2.4899	2.6918	2.8772
	6	0.1991	0.3963	0.5897	0.7779	0.9596	1.1343	1.3014	1.4609	1.6127	1.7572	2.0253	2.2682	2.4891	2.6909	2.8762
	7	0.1991	0.3962	0.5895	0.7777	0.9594	1.1341	1.3011	1.4605	1.6123	1.7568	2.0248	2.2676	2.4884	2.6901	2.8753
	8	0.1990	0.3961	0.5894	0.7775	0.9592	1.1338	1.3009	1.4602	1.6120	1.7563	2.0243	2.2670	2.4877	2.6894	2.8745
	9	0.1990	0.3960	0.5893	0.7774	0.9590	1.1336	1.3006	1.4599	1.6116	1.7560	2.0238	2.2665	2.4871	2.6887	2.8737
	1	0.1989	0.3959	0.5892	0.7772	0.9588	1.1334	1.3003	1.4596	1.6113	1.7556	2.0234	2.2660	2.4865	2.6880	2.8730
	2	0.1987	0.3954	0.5883	0.7760	0.9574	1.1316	1.2983	1.4572	1.6086	1.7526	2.0198	2.2618	2.4818	2.6827	2.8671
	3	0.1984	0.3949	0.5876	0.7751	0.9562	1.1302	1.2967	1.4554	1.6066	1.7504	2.0171	2.2587	2.4782	2.6786	2.8625
	4	0.1982	0.3945	0.5871	0.7744	0.9553	1.1291	1.2953	1.4539	1.6049	1.7485	2.0148	2.2560	2.4751	2.3752	2.8587
10^{-5}	5	0.1981	0.3942	0.5866	0.7737	0.9544	1.1281	1.2941	1.4526	1.6034	1.7468	2.0128	2.2536	2.4724	2.6721	2.8553
	6	0.1979	0.3939	0.5861	0.7731	0.9537	1.1271	1.2931	1.4513	1.6020	1.7452	2.0110	2.2515	2.4700	2.6694	2.8523
	7	0.1978	0.3936	0.5857	0.7725	0.9530	1.1263	1.2921	1.4502	1.6007	1.7438	2.0093	2.2495	2.4677	2.6669	2.8495
	8	0.1976	0.3933	0.5853	0.7720	0.9523	1.1255	1.2912	1.4492	1.5996	1.7425	2.0077	2.2477	2.4657	2.6645	2.8469
	9	0.1975	0.3931	0.5849	0.7715	0.9517	1.1248	1.2903	1.4482	1.5984	1.7413	2.0062	2.2460	2.4637	2.6623	2.8444
	1	0.1974	0.3929	0.5846	0.7710	0.9511	1.1241	1.2895	1.4473	1.5974	1.7402	2.0049	2.2444	2.4619	2.6603	2.8421
	2	0.1965	0.3910	0.5818	0.7673	0.9465	1.1185	1.2830	1.4398	1.5890	1.7308	1.9936	2.2313	2.4469	2.6434	2.8234
	3	0.1958	0.3896	0.5796	0.7644	0.9429	1.1142	1.2780	1.4341	1.5825	1.7236	1.9850	2.2212	2.4354	2.6305	2.8091
	4	0.1952	0.3883	0.5778	0.7620	0.9398	1.1106	1.2737	1.4292	1.5771	1.7176	1.9778	2.2128	2.4258	2.6197	2.7970
10^{-4}	5	0.1946	0.3873	0.5762	0.7599	0.9372	1.1074	1.2700	1.4250	1.5723	1.7123	1.9714	2.2053	2.4172	2.6101	2.7864
	6	0.1941	0.3863	0.5748	0.7580	0.9348	1.1045	1.2666	1.4211	1.5680	1.7075	1.9656	2.1986	2.4095	2.6014	2.7768
	7	0.1937	0.3854	0.5734	0.7562	0.9326	1.1018	1.2635	1.4176	1.5640	1.7030	1.9603	2.1924	2.4025	2.5934	2.7679
	8	0.1933	0.3846	0.5722	0.7545	0.9305	1.0994	1.2607	1.4143	1.5603	1.6989	1.9554	2.1866	2.3959	2.5860	2.7597
	9	0.1929	0.3838	0.5710	0.7530	0.9286	1.0970	1.2579	1.4112	1.5568	1.6951	1.9507	2.1812	2.3897	2.5791	2.7519
	1	0.1925	0.3831	0.5699	0.7515	0.9267	1.0948	1.2554	1.4083	1.5535	1.6914	1.9463	2.1761	2.3838	2.5725	2.7446
	2	0.1896	0.3772	0.5611	0.7397	0.9120	1.0771	1.2347	1.3846	1.5270	1.6619	1.9109	2.1348	2.3367	2.5195	2.6857
	3	0.1873	0.3727	0.5543	0.7307	0.9007	1.0636	1.2189	1.3666	1.5066	1.6393	1.8838	2.1032	2.3006	2.4788	2.6406
	4	0.1854	0.3689	0.5486	0.7231	0.8912	1.0521	1.2056	1.3513	1.4895	1.6203	1.8610	2.0766	2.2702	2.4447	2.6027
10^{-3}	5	0.1837	0.3655	0.5435	0.7163	0.8828	1.0421	1.1938	1.3379	1.4744	1.6035	1.8409	2.0532	2.2434	2.4146	2.5693
	6	0.1822	0.3625	0.5390	0.7103	0.8752	1.0330	1.1832	1.3258	1.4608	1.5884	1.8228	2.0320	2.2193	2.3875	2.5393
	7	0.1808	0.3597	0.5348	0.7047	0.8682	1.0246	1.1735	1.3147	1.4483	1.5745	1.8061	2.0126	2.1972	2.3626	2.5117
	8	0.1795	0.3571	0.5310	0.6995	0.8618	1.0169	1.1645	1.3044	1.4367	1.5616	1.7907	1.9946	2.1766	2.3395	2.4861
	9	0.1783	0.3547	0.5273	0.6947	0.8557	1.0096	1.1560	1.2947	1.4258	1.5495	1.7762	1.9777	2.1573	2.3179	2.4620
	1	0.1772	0.3524	0.5239	0.6901	0.8500	1.0027	1.1480	1.2855	1.4155	1.5381	1.7625	1.9617	2.1391	2.2975	2.4394
	2	0.1680	0.3340	0.4962	0.6533	0.8040	0.9476	1.0836	1.2121	1.3329	1.4464	1.6527	1.8340	1.9935	2.1342	2.2587
	3	0.1610	0.3200	0.4753	0.6253	0.7691	0.9057	1.0349	1.1564	1.2703	1.3770	1.5697	1.7376	1.8839	2.0116	2.1233
	4	0.1551	0.3083	0.4578	0.6020	0.7400	0.8708	0.9942	1.1100	1.2183	1.3193	1.5008	1.6577	1.7932	1.9103	2.0117
10^{-2}	5	0.1500	0.2981	0.4425	0.5817	0.7146	0.8404	0.9588	1.0696	1.1730	1.2691	1.4410	1.5884	1.7147	1.8229	1.9156
	6	0.1455	0.2890	0.4289	0.5635	0.6919	0.8132	0.9272	1.0336	1.1326	1.2243	1.3877	1.5268	1.6450	1.7454	1.8307
	7	0.1413	0.2807	0.4164	0.5470	0.6713	0.7885	0.8994	1.0008	1.0958	1.1837	1.3394	1.4711	1.5821	1.6756	1.7543
	8	0.1375	0.2731	0.4050	0.5317	0.6522	0.7658	0.8720	0.9707	1.0621	1.1464	1.2951	1.4200	1.5246	1.6120	1.6848
	9	0.1339	0.2660	0.3943	0.5176	0.6346	0.7447	0.8474	0.9428	1.0308	1.1118	1.2541	1.3729	1.4716	1.5534	1.6210
	1	0.130	0.2593	0.3844	0.5043	0.6181	0.7249	0.8245	0.9167	1.0016	1.0795	1.2159	1.3290	1.4223	1.4991	1.5619
	2	0.1051	0.2084	0.3081	0.4030	0.4920	0.5744	0.6500	0.7186	0.7806	0.8362	0.9297	1.0029	1.0595	1.1026	1.1352
	3	8.74(-2)	0.1731	0.2554	0.3331	0.4053	0.4713	0.5309	0.5842	0.6313	0.6727	0.7400	0.7899	0.8261	0.8519	0.8699
	4	7.39(-2)	0.1462	0.2153	0.2801	0.3397	0.3935	0.4415	0.4837	0.5203	0.5519	0.6015	0.6363	0.6602	0.6760	0.6863
10^{-1}	5	6.32(-2)	0.1248	0.1835	0.2381	0.2878	0.3323	0.3714	0.4052	0.4341	0.4584	0.4955	0.5203	0.5362	0.5462	0.5521
	6	5.44(-2)	0.1074	0.1575	0.2039	0.2458	0.2828	0.3149	0.3423	0.3652	0.3842	0.4122	0.4300	0.4408	0.4471	0.4506
	7	4.71(-2)	9.29(-2)	0.1360	0.1756	0.2111	0.2421	0.2686	0.2909	0.3093	0.3242	0.3455	0.3583	0.3657	0.3698	0.3719
	8	4.10(-2)	8.06(-2)	0.1179	0.1519	0.1821	0.2082	0.2302	0.2484	0.2632	0.2750	0.2913	0.3007	0.3058	0.3084	0.3096
	9	3.57(-2)	7.03(-2)	0.1026	0.1319	0.1576	0.1797	0.1980	0.2130	0.2250	0.2343	0.2468	0.2537	0.2572	0.2589	0.2597
	1	3.13(-2)	6.14(-2)	8.95(-2)	0.1148	0.1369	0.1555	0.1709	0.1833	0.1929	0.2004	0.2101	0.2151	0.2175	0.2186	0.2191
	2	9.01(-3)	1.75(-2)	2.51(-2)	3.16(-2)	3.67(-2)	4.07(-2)	4.35(-2)	4.55(-2)	4.69(-2)	4.77(-2)	4.85(-2)	4.88(-2)			4.88(-2)
	3	2.82(-3)	5.44(-3)	7.68(-3)	9.47(-3)	1.08(-2)	1.17(-2)	1.23(-2)	1.26(-2)	1.28(-2)	1.30(-2)					1.30(-2)
	4	9.20(-4)	1.76(-3)	2.44(-3)	2.96(-3)	3.31(-3)	3.53(-3)	3.66(-3)	3.72(-3)	3.76(-3)	3.77(-3)					3.77(-3)
1	5	3.07(-4)	5.80(-4)	7.96(-4)	9.49(-4)	1.05(-3)	1.10(-3)	1.13(-3)	1.14(-3)	1.15(-3)						1.15(-3)
	6	1.04(-4)	1.95(-4)	2.64(-4)	3.10(-4)	3.36(-4)	3.50(-4)	3.56(-4)	3.59(-4)	3.60(-4)						3.60(-4)
	7	3.56(-5)	6.61(-5)	8.84(-5)	1.02(-4)	1.10(-4)	1.13(-4)	1.15(-4)	1.15(-4)	1.15(-4)						1.15(-4)
	8	1.23(-5)	2.26(-5)	2.99(-5)	3.42(-5)	3.63(-5)	3.72(-5)	3.75(-5)	3.76(-5)	3.77(-5)						3.77(-5)
	9	4.28(-6)	7.79(-6)	1.02(-5)	1.15(-5)	1.21(-5)	1.23(-5)	1.24(-5)	1.24(-5)							1.24(-5)
	10	1.49(-6)	2.70(-6)	3.48(-6)	3.90(-6)	4.07(-6)	4.13(-6)	4.15(-6)	4.16(-6)							4.16(-6)

Values of M(u, β) are equal to W(u) for u greater than 10 and all values of β

The numbers in parentheses are powers of 10 by which the other numbers are multiplied e.g., 8.74(-2)=0.087

TABLE IV (*Continued*)

u	β	2.0	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0
	0	2.8872	3.0593	3.2188	3.3675	3.5064	3.6369	3.7597	3.8757	3.9856	4.0900	4.1894	4.2842	4.3748	4.4616	4.5448	4.6248	4.7018	4.7760	4.8475	4.9167	4.9835
10^{-6}	1	2.8827	3.0543	3.2134	3.3616	3.5001	3.6301	3.7525	3.8681	3.9775	4.0815	4.1804	4.2747	4.3649	4.4512	4.5340	4.6136	4.6901	4.7638	4.8349	4.9036	4.9700
	2	2.8809	3.0523	3.2112	3.3592	3.4975	3.6273	3.7495	3.8649	3.9742	4.0779	4.1766	4.2708	4.3608	4.4469	4.5295	4.6089	4.6852	4.7588	4.8297	4.8982	4.9644
	3	2.8794	3.0507	3.2095	3.3573	3.4955	3.6251	3.7472	3.8625	3.9716	4.0752	4.1738	4.2678	4.3576	4.4436	4.5261	4.6053	4.6815	4.7549	4.8257	4.8940	4.9601
	4	2.8782	3.0494	3.2080	3.3557	3.4938	3.6233	3.7453	3.8604	3.9694	4.0729	4.1714	4.2653	4.3550	4.4408	4.5232	4.6023	4.6784	4.7516	4.8223	4.8905	4.9565
	5	2.8772	3.0482	3.2067	3.3544	3.4923	3.6217	3.7436	3.8586	3.9675	4.0709	4.1692	4.2630	4.3526	4.4384	4.5206	4.5996	4.6756	4.7488	4.8193	4.8874	4.9533
	6	2.8762	3.0471	3.2056	3.3531	3.4910	3.6203	3.7420	3.8569	3.9658	4.0690	4.1673	4.2610	4.3505	4.4362	4.5183	4.5972	4.6731	4.7462	4.8166	4.8846	4.9504
	7	2.8753	3.0462	3.2045	3.3519	3.4897	3.6190	3.7406	3.8554	3.9642	4.0674	4.1655	4.2591	4.3486	4.4341	4.5162	4.5950	4.6708	4.7438	4.8141	4.8821	4.9477
	8	2.8745	3.0453	3.2035	3.3509	3.4886	3.6177	3.7393	3.8540	3.9627	4.0658	4.1639	4.2574	4.3467	4.4323	4.5142	4.5929	4.6686	4.7415	4.8118	4.8797	4.9452
	9	2.8737	3.0444	3.2026	3.3499	3.4875	3.6166	3.7380	3.8527	3.9613	4.0643	4.1623	4.2558	4.3450	4.4305	4.5124	4.5910	4.6666	4.7394	4.8097	4.8774	4.9429
10^{-5}	1	2.8730	3.0436	3.2017	3.3489	3.4865	3.6155	3.7369	3.8515	3.9600	4.0629	4.1609	4.2542	4.3434	4.4288	4.5106	4.5892	4.6647	4.7375	4.8076	4.8753	4.9407
	2	2.8671	3.0371	3.1946	3.3412	3.4782	3.6066	3.7274	3.8414	3.9493	4.0517	4.1490	4.2418	4.3304	4.4152	4.4964	4.5744	4.6494	4.7215	4.7911	4.8582	4.9230
	3	2.8625	3.0321	3.1892	3.3353	3.4718	3.5998	3.7202	3.8337	3.9412	4.0431	4.1400	4.2323	4.3204	4.4048	4.4855	4.5631	4.6376	4.7093	4.7784	4.8450	4.9094
	4	2.8587	3.0279	3.1846	3.3304	3.4665	3.5941	3.7140	3.8272	3.9343	4.0358	4.1323	4.2243	4.3120	4.3960	4.4764	4.5535	4.6276	4.6989	4.7677	4.8339	4.8979
	5	2.8553	3.0242	3.1806	3.3260	3.4618	3.5890	3.7087	3.8215	3.9282	4.0294	4.1256	4.2172	4.3046	4.3882	4.4683	4.5451	4.6189	4.6899	4.7582	4.8242	4.8878
	6	2.8523	3.0209	3.1769	3.3220	3.4575	3.5844	3.7038	3.8163	3.9227	4.0236	4.1195	4.2108	4.2979	4.3812	4.4610	4.5375	4.6110	4.6816	4.7497	4.8153	4.8787
	7	2.8495	3.0178	3.1735	3.3184	3.4536	3.5802	3.6993	3.8116	3.9177	4.0183	4.1139	4.2049	4.2918	4.3748	4.4543	4.5305	4.6037	4.6741	4.7419	4.8072	4.8703
	8	2.8469	3.0149	3.1704	3.3150	3.4499	3.5763	3.6951	3.8071	3.9130	4.0133	4.1087	4.1994	4.2860	4.3688	4.4480	4.5240	4.5969	4.6670	4.7346	4.7997	4.8625
	9	2.8444	3.0122	3.1675	3.3118	3.4465	3.5727	3.6912	3.8030	3.9086	4.0087	4.1038	4.1943	4.2806	4.3632	4.4421	4.5178	4.5905	4.6604	4.7277	4.7926	4.8551
10^{-4}	1	2.8421	3.0097	3.1647	3.3088	3.4433	3.5692	3.6875	3.7990	3.9044	4.0043	4.0992	4.1894	4.2756	4.3578	4.4366	4.5121	4.5845	4.6542	4.7212	4.7859	4.8482
	2	2.8234	2.9891	3.1423	3.2845	3.4171	3.5412	3.6576	3.7673	3.8708	3.9688	4.0618	4.1502	4.2345	4.3149	4.3918	4.4654	4.5360	4.6038	4.6690	4.7317	4.7922
	3	2.8091	2.9733	3.1251	3.2659	3.3970	3.5197	3.6347	3.7429	3.8450	3.9416	4.0332	4.1202	4.2030	4.2820	4.3574	4.4296	4.4988	4.5652	4.6289	4.6903	4.7493
	4	2.7970	2.9600	3.1106	3.2502	3.3801	3.5015	3.6154	3.7224	3.8233	3.9187	4.0090	4.0948	4.1764	4.2542	4.3285	4.3995	4.4674	4.5326	4.5952	4.6553	4.7132
	5	2.7864	2.9483	3.0978	3.2363	3.3652	3.4856	3.5984	3.7043	3.8042	3.8985	3.9878	4.0725	4.1531	4.2298	4.3030	4.3729	4.4399	4.5040	4.5655	4.6246	4.6814
	6	2.7768	2.9378	3.0863	3.2238	3.3518	3.4712	3.5830	3.6880	3.7869	3.8802	3.9686	4.0524	4.1320	4.2077	4.2800	4.3490	4.4150	4.4781	4.5387	4.5969	4.6527
	7	2.7679	2.9280	3.0757	3.2123	3.3394	3.4579	3.5688	3.6730	3.7710	3.8635	3.9509	4.0338	4.1126	4.1875	4.2588	4.3270	4.3921	4.4544	4.5141	4.5714	4.6264
	8	2.7597	2.9190	3.0658	3.2017	3.3279	3.4456	3.5557	3.6590	3.7562	3.8479	3.9345	4.0166	4.0945	4.1686	4.2392	4.3065	4.3708	4.4323	4.4912	4.5477	4.6019
	9	2.7519	2.9105	3.0565	3.1916	3.3171	3.4340	3.5434	3.6459	3.7423	3.8332	3.9191	4.0004	4.0776	4.1509	4.2207	4.2873	4.3508	4.4116	4.4697	4.5255	4.5789
10^{-3}	1	2.7448	2.9024	3.0478	3.1821	3.3069	3.4231	3.5317	3.6335	3.7292	3.8194	3.9046	3.9852	4.0616	4.1342	4.2033	4.2691	4.3320	4.3920	4.4494	4.5045	4.5572
	2	2.6857	2.8377	2.9771	3.1056	3.2245	3.3349	3.4377	3.5337	3.6236	3.7080	3.7874	3.8623	3.9329	3.9998	4.0632	4.1233	4.1805	4.2349	4.2867	4.3360	4.3832
	3	2.6406	2.7881	2.9231	3.0471	3.1616	3.2675	3.3659	3.4575	3.5430	3.6230	3.6981	3.7686	3.8350	3.8975	3.9566	4.0125	4.0654	4.1156	4.1632	4.2084	4.2514
	4	2.6027	2.7464	2.8776	2.9980	3.1087	3.2110	3.3056	3.3936	3.4754	3.5518	3.6233	3.6902	3.7530	3.8120	3.8676	3.9199	3.9694	4.0161	4.0602	4.1020	4.1416
	5	2.5693	2.7098	2.8377	2.9548	3.0623	3.1613	3.2528	3.3375	3.4162	3.4894	3.5577	3.6215	3.6813	3.7372	3.7898	3.8391	3.8856	3.9293	3.9705	4.0094	4.0462
	6	2.5393	2.6767	2.8018	2.9159	3.0205	3.1166	3.2052	3.2871	3.3629	3.4334	3.4989	3.5599	3.6169	3.6702	3.7200	3.7667	3.8105	3.8517	3.8903	3.9267	3.9609
	7	2.5117	2.6464	2.7688	2.8802	2.9822	3.0757	3.1616	3.2409	3.3142	3.3821	3.4451	3.5036	3.5581	3.6090	3.6564	3.7007	3.7422	3.7810	3.8174	3.8515	3.8835
	8	2.4861	2.6183	2.7382	2.8472	2.9466	3.0377	3.1213	3.1982	3.2691	3.3346	3.3953	3.4516	3.5038	3.5524	3.5977	3.6398	3.6792	3.7159	3.7502	3.7823	3.8123
	9	2.4620	2.5920	2.7095	2.8162	2.9134	3.0022	3.0835	3.1582	3.2270	3.2903	3.3489	3.4030	3.4532	3.4998	3.5430	3.5832	3.6206	3.6554	3.6878	3.7181	3.7463
10^{-2}	1	2.4394	2.5671	2.6825	2.7870	2.8820	2.9687	3.0480	3.1206	3.1873	3.2487	3.3052	3.3574	3.4057	3.4503	3.4917	3.5300	3.5656	3.5987	3.6294	3.6580	3.6845
	2	2.2587	2.3692	2.4675	2.5552	2.6337	2.7041	2.7673	2.8243	2.8756	2.9218	2.9637	3.0015	3.0357	3.0666	3.0946	3.1200	3.1430	3.1638	3.1827	3.1998	3.2153
	3	2.1233	2.2212	2.3072	2.3830	2.4498	2.5088	2.5610	2.6073	2.6482	2.6846	2.7168	2.7453	2.7707	2.7932	2.8131	2.8307	2.8464	2.8602	2.8724	2.8832	2.8928
	4	2.0117	2.0996	2.1759	2.2423	2.3000	2.3503	2.3942	2.4324	2.4658	2.4949	2.5202	2.5423	2.5615	2.5782	2.5927	2.6052	2.6161	2.6256	2.6337	2.6408	2.6468
	5	1.9156	1.9951	2.0634	2.1221	2.1725	2.2158	2.2531	2.2851	2.3125	2.3361	2.3563	2.3735	2.3883	2.4009	2.4116	2.4207	2.4284	2.4350	2.4405	2.4452	2.4491
	6	1.8307	1.9031	1.9645	2.0167	2.0610	2.0986	2.1305	2.1574	2.1802	2.1995	2.2157	2.2294	2.2408	2.2504	2.2584	2.2651	2.2706	2.2752	2.2790	2.2821	2.2846
	7	1.7543	1.8205	2.8760	1.9227	1.9618	1.9946	2.0220	2.0449	2.0640	2.0798	2.0929	2.1038	2.1127	2.1201	2.1261	2.1310	2.1350	2.1382	2.1408	2.1429	2.1446
	8	1.6848	1.7455	1.7959	1.8378	1.8725	1.9012	1.9249	1.9444	1.9604	1.9734	1.9841	1.9928	1.9998	2.0055	2.0101	2.0137	2.0166	2.0189	2.0207	2.0221	2.0233
	9	1.6210	1.6767	1.7226	1.7603	1.7912	1.8164	1.8370	1.8536	1.8671	1.8780	1.8867	1.8937	1.8992	1.9036	1.9071	1.9098	1.9119	1.9135	1.9148	1.9158	1.9165
10^{-1}	1	1.5619	1.6133	1.6552	1.6892	1.7167	1.7389	1.7568	1.7711	1.7825	1.7915	1.7987	1.8043	1.8087	1.8121	1.8147	1.8168	1.8183	1.8195	1.8204	1.8211	1.8216
	2	1.1352	1.1596	1.1777	1.1909	1.2004	1.2073	1.2122	1.2156	1.2179	1.2195	1.2206	1.2213	1.2218	1.2221	1.2223	1.2224	1.2225	1.2226	1.2226	1.2226	1.2226
	3	0.8699	0.8823	0.8907	0.8962	0.8998	0.9021	0.9035	0.9044	0.9049	0.9053	0.9054	0.9056	0.9056	0.9056	0.9057						0.9057
	4	0.6863	0.6928	0.6968	0.6992	0.7006	0.7014	0.7019	0.7021	0.7023	0.7023	0.7024										0.7024
	5	0.5521	0.5556	0.5576	0.5587	0.5592	0.5595	0.5597	0.5597	0.5597												0.5597
	6	0.4506	0.4525	0.4535	0.4540	0.4542	0.4543	0.4543														0.4543
	7	0.3719	0.3729	0.3734	0.3736	0.3737	0.3737															0.3737
	8	0.3096	0.3102	0.3104	0.3105																	0.3105
	9	0.2597	0.2600	0.2601	0.2602																	0.2602
	1	0.2191	0.2193	0.2194																		0.2194

Values of $M(u,\beta)$ are equal to $W(u)$ for u greater than 1 and all values of β

u	β	6.0	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8	8.0	8.2	8.4	8.6	8.8	9.0	9.2	9.4	9.6	9.8	10
	0	4.9835	5.0482	5.1109	5.1718	5.2308	5.2882	5.3440	5.3983	5.4511	5.5026	5.5529	5.6019	5.6497	5.6965	5.7421	5.7868	5.8305	5.8733	5.9151	5.9562	5.9964
10^{-6}	1	4.9700	5.0343	5.0965	5.1569	5.2155	5.2724	5.3278	5.3816	5.4340	5.4851	5.5349	5.5834	5.6308	5.6771	5.7223	5.7666	5.8098	5.8521	5.8935	5.9341	5.9739
	2	4.9644	5.0285	5.0905	5.1507	5.2091	5.2659	5.3210	5.3747	5.4269	5.4778	5.5274	5.5758	5.6230	5.6691	5.7141	5.7581	5.8012	5.8433	5.8846	5.9250	5.9645
	3	4.9601	5.0240	5.0860	5.1460	5.2043	5.2609	5.3159	5.3694	5.4215	5.4722	5.5217	5.5699	5.6170	5.6629	5.7078	5.7517	5.7946	5.8366	5.8777	5.9179	5.9573
	4	4.9565	5.0203	5.0821	5.1420	5.2002	5.2566	5.3115	5.3649	5.4169	5.4675	5.5168	5.5649	5.6119	5.6577	5.7025	5.7463	5.7890	5.8309	5.8719	5.9120	5.9513
	5	4.9533	5.0170	5.0787	5.1385	5.1965	5.2529	5.3077	5.3610	5.4128	5.4633	5.5126	5.5606	5.6074	5.6531	5.6978	5.7415	5.7841	5.8259	5.8668	5.9068	5.9460
	6	4.9504	5.0140	5.0756	5.1353	5.1933	5.2495	5.3042	5.3574	5.4092	5.4596	5.5087	5.5566	5.6034	5.6490	5.6936	5.7371	5.7797	5.8214	5.8621	5.9021	5.9412
	7	4.9477	5.0112	5.0728	5.1324	5.1902	5.2464	5.3010	5.3541	5.4058	5.4561	5.5052	5.5530	5.5996	5.6452	5.6897	5.7331	5.7756	5.8172	5.8579	5.8977	5.9367
	8	4.9452	5.0087	5.0701	5.1297	5.1874	5.2435	5.2981	5.3511	5.4027	5.4529	5.5019	5.5496	5.5962	5.6416	5.6860	5.7294	5.7718	5.8133	5.8539	5.8937	5.9326
	9	4.9429	5.0063	5.0676	5.1271	5.1848	5.2408	5.2953	5.3482	5.3997	5.4499	5.4988	5.5464	5.5929	5.6383	5.6826	5.7259	5.7683	5.8097	5.8502	5.8899	5.9287
10^{-5}	1	4.9407	5.0040	5.0653	5.1247	5.1823	5.2383	5.2926	5.3455	5.3969	5.4470	5.4958	5.5434	5.5898	5.6352	5.6794	5.7226	5.7649	5.8063	5.8467	5.8863	5.9251
	2	4.9230	4.9857	5.0464	5.1052	5.1622	5.2176	5.2714	5.3236	5.3745	5.4240	5.4722	5.5192	5.5650	5.6097	5.6534	5.6961	5.7377	5.7785	5.8183	5.8573	5.8955
	3	4.9094	4.9716	5.0319	5.0902	5.1468	5.2017	5.2550	5.3069	5.3573	5.4063	5.4541	5.5006	5.5460	5.5903	5.6335	5.6757	5.7169	5.7572	5.7966	5.8351	5.8729
	4	4.8979	4.9598	5.0196	5.0776	5.1338	5.1883	5.2413	5.2927	5.3427	5.3914	5.4388	5.4849	5.5299	5.5738	5.6167	5.6585	5.6993	5.7392	5.7782	5.8164	5.8538
	5	4.8878	4.9493	5.0089	5.0665	5.1224	5.1766	5.2292	5.2803	5.3299	5.3783	5.4253	5.4711	5.5158	5.5594	5.6018	5.6433	5.6838	5.7234	5.7621	5.7999	5.8369
	6	4.8787	4.9399	4.9991	5.0565	5.1120	5.1659	5.2182	5.2690	5.3184	5.3664	5.4131	5.4587	5.5030	5.5463	5.5885	5.6296	5.6698	5.7091	5.7475	5.7850	5.8217
	7	4.8703	4.9312	4.9902	5.0472	5.1025	5.1561	5.2081	5.2586	5.3077	5.3555	5.4020	5.4472	5.4913	5.5342	5.5762	5.6171	5.6570	5.6960	5.7341	5.7713	5.8078
	8	4.8625	4.9232	4.9818	5.0386	5.0937	5.1470	5.1988	5.2490	5.2979	5.3453	5.3915	5.4365	5.4803	5.5231	5.5647	5.6053	5.6450	5.6837	5.7216	5.7586	5.7948
	9	4.8551	4.9156	4.9740	5.0306	5.0853	5.1384	5.1900	5.2400	5.2886	5.3358	5.3818	5.4265	5.4701	5.5126	5.5540	5.5943	5.6338	5.6723	5.7099	5.7466	5.7826
10^{-4}	1	4.8482	4.9084	4.9686	5.0229	5.0775	5.1303	5.1816	5.2314	5.2798	5.3268	5.3725	5.4170	5.4604	5.5026	5.5438	5.5840	5.6231	5.6614	5.6988	5.7353	5.7710
	2	4.7922	4.8506	4.9069	4.9614	5.0141	5.0651	5.1145	5.1624	5.2089	5.2541	5.2980	5.3406	5.3821	5.4225	5.4619	5.5002	5.5375	5.5739	5.6095	5.6441	5.6780
	3	4.7493	4.8062	4.8612	4.9142	4.9655	5.0151	5.0631	5.1096	5.1547	5.1985	5.2409	5.2822	5.3223	5.3612	5.3992	5.4361	5.4720	5.5070	5.5411	5.5744	5.6069
	4	4.7132	4.7689	4.8227	4.8745	4.9246	4.9730	5.0198	5.0652	5.1091	5.1518	5.1929	5.2330	5.2719	5.3097	5.3464	5.3822	5.4169	5.4508	5.4837	5.5158	5.5471
	5	4.3814	4.7361	4.7888	4.8396	4.8886	4.9360	4.9818	5.0261	5.0689	5.1105	5.1507	5.1898	5.2276	5.2644	5.3001	5.3348	5.3686	5.4014	5.4333	5.4644	5.4947
	6	4.6527	4.7065	4.7582	4.8081	4.8562	4.9026	4.9475	4.9908	5.0327	5.0733	5.1127	5.1508	5.1877	5.2236	5.2583	5.2921	5.3249	5.3568	5.3879	5.4180	5.4474
	7	4.6264	4.6793	4.7301	4.7791	4.8264	4.8719	4.9159	4.9584	4.9995	5.0393	5.0777	5.1150	5.1511	5.1861	5.2200	5.2530	5.2850	5.3160	5.3462	5.3755	5.4041
	8	4.6019	4.6540	4.7040	4.7522	4.7987	4.8435	4.8867	4.9284	4.9687	5.0076	5.0453	5.0818	5.1171	5.1513	5.1845	5.2166	5.2478	5.2781	5.3075	5.3361	5.3639
	9	4.5789	4.6302	4.6796	4.7270	4.7727	4.8168	4.8592	4.9002	4.9397	4.9780	5.0149	5.0506	5.0852	5.1187	5.1512	5.1826	5.2131	5.2427	5.2714	5.2992	5.3263
10^{-3}	1	4.5572	4.6078	4.6565	4.7032	4.7482	4.7915	4.8333	4.8736	4.9124	4.9500	4.9862	5.0213	5.0552	5.0880	5.1197	5.1505	5.1803	5.2092	5.2372	5.2644	5.2908
	2	4.3832	4.4282	4.4713	4.5125	4.5519	4.5898	4.6260	4.6609	4.6943	4.7264	4.7573	4.7870	4.8155	4.8430	4.8695	4.8950	4.9196	4.9433	4.9662	4.9882	5.0095
	3	4.2514	4.2923	4.3312	4.3684	4.4038	4.4376	4.4699	4.5007	4.5302	4.5584	4.5854	4.6113	4.5360	4.6597	4.6824	4.7042	4.7251	4.7452	4.7644	4.7829	4.8006
	4	4.1416	4.1792	4.2148	4.2487	4.2808	4.3114	4.3405	4.3682	4.3945	4.4197	4.4436	4.4664	4.4881	4.5089	4.5287	4.5476	4.5656	4.5829	4.5993	4.6150	4.6301
	5	4.0462	4.0809	4.1137	4.1448	4.1742	4.2021	4.2285	4.2535	4.2773	4.2998	4.3212	4.3415	4.3608	4.3792	4.3966	4.4131	4.4288	4.4438	4.4580	4.4716	4.4844
	6	3.9609	3.9932	4.0236	4.0523	4.0793	4.1048	4.1290	4.1517	4.1733	4.1936	4.2129	4.2311	4.2483	4.2646	4.2800	4.2946	4.3084	4.3214	4.3338	4.3456	4.3566
	7	3.8835	3.9136	3.9418	3.9684	3.9934	4.0168	4.0390	4.0598	4.0794	4.0978	4.1152	4.1316	4.1470	4.1616	4.1753	4.1882	4.2004	4.2119	4.2227	4.2329	4.2425
	8	3.8123	3.8404	3.8668	3.8914	3.9146	3.9362	3.9565	3.9756	3.9935	4.0103	4.0261	4.0409	4.0548	4.0678	4.0801	4.0916	4.1024	4.1125	4.1220	4.1309	4.1393
	9	3.7463	3.7726	3.7972	3.8202	3.8416	3.8617	3.8805	3.8980	3.9144	3.9297	3.9440	3.9575	3.9700	3.9817	3.9927	4.0029	4.0125	4.0215	4.0299	4.0377	4.0450
10^{-2}	1	3.6845	3.7093	3.7323	3.7537	3.7737	3.7923	3.8096	3.8258	3.8408	3.8548	3.8679	3.8801	3.8914	3.9020	3.9119	3.9210	3.9296	3.9375	3.9449	3.9518	3.9582
	2	3.2153	3.2293	3.2419	3.2534	3.2638	3.2731	3.2816	3.2892	3.2961	3.3023	3.3079	3.3130	3.3175	3.3215	3.3252	3.3285	3.3314	3.3340	3.3364	3.3385	3.3403
	3	2.8928	2.9012	2.9086	2.9151	2.9209	2.9259	2.9303	2.9342	2.9376	2.9405	2.9431	2.9453	2.9473	2.9489	2.9504	2.9517	2.9528	2.9537	2.9545	2.9552	2.9558
	4	2.6468	2.6520	2.6565	2.6603	2.6636	2.6664	2.6688	2.6708	2.6725	2.6740	2.6752	2.6762	2.6771	2.6778	2.6784	2.6789	2.6793	2.6797	2.6800	2.6802	2.6804
	5	2.4491	2.4523	2.4551	2.4574	2.4593	2.4609	2.4622	2.4632	2.4641	2.4648	2.4654	2.4659	2.4663	2.4666	2.4669	2.4671	2.4673	2.4674	2.4675	2.4676	2.4677
	6	2.2846	2.2867	2.2884	2.2898	2.2909	2.2918	2.2926	2.2931	2.2936	2.2940	2.2943	2.2945	2.2947	2.2948	2.2949	2.2950	2.2951	2.2951	2.2952	2.2952	2.2952
	7	2.1446	2.1460	2.1470	2.1479	2.1486	2.1491	2.1495	2.1498	2.1500	2.1502	2.1504	2.1505	2.1506	2.1506	2.1507	2.1507	2.1508				2.1508
	8	2.0233	2.0241	2.0248	2.0253	2.0257	2.0260	2.0263	2.0264	2.0266	2.0257	2.0267	2.0268	2.0268	2.0259							2.0269
	9	1.9155	1.9171	1.9175	1.9178	1.9181	1.9183	1.9184	1.9185	1.9186	1.9186	1.9185										1.9186
10^{-1}	1	1.8216	1.8219	1.8222	1.8224	1.8226	1.8227	1.8227	1.8228	1.8228	1.8229											1.8229
	2	1.2226	1.2226	1.2226																		1.2226

Values of $M(u,\beta)$ are equal to $W(u)$ for u greater than $2 \cdot 10^{-1}$ and all values of β

TABLE IV (*Continued*)

u	β	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50
	0	5.9964	6.3595	6.6668	6.9333	7.1684	7.3789	7.5692	7.7431	7.9030	8.0511	8.1890	8.3180	8.4392	8.5535	8.6615	8.7641	8.8616	8.9546	9.0435	9.1286	9.2102
10^{-6}	1	5.9739	6.3325	6.6353	6.8973	7.1279	7.3339	7.5197	7.6891	7.8445	7.9881	8.1215	8.2460	8.3627	8.4725	8.5761	8.6741	8.7671	8.8556	8.9400	9.0206	9.0977
	2	5.9645	6.3213	6.6223	6.8823	7.1111	7.3152	7.4992	7.6667	7.8202	7.9620	8.0935	8.2161	8.3309	8.4388	8.5406	8.6367	8.7279	8.8145	8.8971	8.9758	9.0510
	3	5.9573	6.3127	6.6122	6.8709	7.0982	7.3008	7.4834	7.6495	7.8016	7.9419	8.0720	8.1932	8.3066	8.4130	8.5133	8.6081	8.6978	8.7830	8.8641	8.9414	9.0152
	4	5.9513	6.3054	6.6038	6.8612	7.0873	7.2887	7.4701	7.6350	7.7859	7.9250	8.0539	8.1739	8.2861	8.3913	8.4904	8.5839	8.6725	8.7565	8.8364	8.9125	8.9851
	5	5.9460	6.2990	6.5963	6.8527	7.0777	7.2781	7.4584	7.6222	7.7721	7.9101	8.0379	8.1569	8.2680	8.3722	8.4702	8.5627	8.6502	8.7331	8.8119	8.8870	8.9586
	6	5.9412	6.2932	6.5896	6.8450	7.0691	7.2685	7.4478	7.6106	7.7596	7.8966	8.0235	8.1415	8.2516	8.3549	8.4519	8.5435	8.6300	8.7120	8.7899	8.8640	8.9346
	7	5.9367	6.2879	6.5834	6.8379	7.0611	7.2596	7.4381	7.6000	7.7481	7.8842	8.0102	8.1273	8.2366	8.3390	8.4352	8.5258	8.6115	8.6926	8.7696	8.8428	8.9126
	8	5.9326	6.2830	6.5776	6.8313	7.0537	7.2514	7.4290	7.5901	7.7374	7.8727	7.9979	8.1142	8.2226	8.3242	8.4196	8.5094	8.5942	8.6745	8.7507	8.8231	8.8921
	9	5.9287	6.2783	6.5722	6.8251	7.0467	7.2436	7.4205	7.5809	7.7273	7.8619	7.9863	8.1018	8.2095	8.3103	8.4049	8.4940	8.5761	8.6576	8.7330	8.8047	8.8729
10^{-5}	1	5.9251	6.2739	6.5671	6.8193	7.0402	7.2363	7.4125	7.5721	7.7178	7.8517	7.9753	8.0901	8.1971	8.2972	8.3910	8.4794	8.5628	8.6416	8.7163	8.7872	8.8547
	2	5.8955	6.2385	6.5257	6.7720	6.9870	7.1773	7.3476	7.5013	7.6412	7.7692	7.8871	7.9960	8.0972	8.1914	8.2795	8.3621	8.4397	8.5127	8.5817	8.6469	8.7087
	3	5.8729	6.2113	6.4940	6.7358	6.9463	7.1321	7.2979	7.4472	7.5826	7.7061	7.8195	7.9240	8.0208	8.1106	8.1943	8.2725	8.3458	8.4145	8.4792	8.5401	8.5976
	4	5.8538	6.1884	6.4673	6.7053	6.9120	7.0940	7.2561	7.4016	7.5332	7.6531	7.7627	7.8636	7.9566	8.0428	8.1229	8.1975	8.2671	8.3322	8.3933	8.4507	8.5047
	5	5.8369	6.1682	6.4438	6.6785	6.8818	7.0606	7.2193	7.3615	7.4899	7.6065	7.7129	7.8105	7.9003	7.9833	8.0602	8.1316	8.1982	8.2602	8.3182	8.3725	8.4235
	6	5.8217	6.1500	6.4225	6.6542	6.8546	7.0303	7.1861	7.3253	7.4508	7.5644	7.6679	7.7626	7.8496	7.9298	8.0038	8.0725	8.1362	8.1955	8.2508	8.3024	8.3507
	7	5.8078	6.1332	6.4030	6.6319	6.8296	7.0025	7.1556	7.2921	7.4149	7.5259	7.6267	7.7188	7.8032	7.8807	7.9522	8.0183	8.0795	8.1364	8.1892	8.2384	8.2843
	8	5.7948	6.1177	6.3848	6.6112	6.8063	6.9767	7.1272	7.2613	7.3815	7.4901	7.5885	7.6781	7.7601	7.8353	7.9044	7.9682	8.0271	8.0817	8.1323	8.1792	8.2229
	9	5.7826	6.1030	6.3678	6.5917	6.7844	6.9525	7.1007	7.2324	7.3503	7.4565	7.5527	7.6401	7.7198	7.7928	7.8597	7.9214	7.9782	8.0306	8.0791	8.1241	8.1658
10^{-4}	1	5.7710	6.0892	6.3517	6.5734	6.7638	6.9296	7.0756	7.2051	7.3208	7.4249	7.5189	7.6042	7.6818	7.7527	7.8177	7.8773	7.9321	7.9826	8.0292	8.0723	8.1122
	2	5.6780	5.9778	6.2221	6.4257	6.5982	6.7463	6.8747	6.9869	7.0856	7.1729	7.2504	7.3194	7.3811	7.4364	7.4861	7.5307	7.5709	7.6072	7.6399	7.6695	7.6962
	3	5.6069	5.8928	6.1233	6.3133	6.4725	6.6075	6.7231	6.8227	6.9091	6.9844	7.0502	7.1080	7.1588	7.2035	7.2429	7.2777	7.3084	7.3356	7.3597	7.3809	7.3997
	4	5.5471	5.8214	6.0406	6.2194	6.3677	6.4920	6.5972	6.6868	6.7635	6.8294	6.8862	6.9353	6.9778	7.0146	7.0465	7.0742	7.0982	7.1191	7.1371	7.1528	7.1663
	5	5.4947	5.7589	5.9681	6.1374	6.2763	6.3915	6.4880	6.5692	6.6378	6.6961	6.7456	6.7878	6.8237	6.8544	6.8805	6.9029	6.9219	6.9380	6.9518	6.9635	6.9734
	6	5.4474	5.7026	5.9031	6.0638	6.1945	6.3019	6.3908	6.4648	6.5266	6.5784	6.6218	6.6583	6.6890	6.7147	6.7363	6.7545	6.7696	6.7823	6.7929	6.8017	6.8090
	7	5.4041	5.6511	5.8437	5.9967	6.1201	6.2205	6.3027	6.3704	6.4263	6.4726	6.5109	6.5427	6.5690	6.5907	6.6087	6.6235	6.6357	6.6457	6.6539	6.6603	6.6661
	8	5.3639	5.6034	5.7887	5.9348	6.0515	6.1456	6.2219	6.2841	6.3349	6.3763	6.4103	6.4380	6.4607	6.4791	6.4942	6.5063	6.5162	6.5241	6.5305	6.5357	6.5397
	9	5.3263	5.5588	5.7374	5.8771	5.9878	6.0762	6.1472	6.2044	6.2506	6.2879	6.3181	6.3424	6.3620	6.3777	6.3903	6.4004	6.4084	6.4147	6.4197	6.4236	6.4267
10^{-3}	1	5.2908	5.5168	5.6892	5.8230	5.9281	6.0113	6.0775	6.1303	6.1724	6.2061	6.2330	6.2543	6.2713	6.2848	6.2954	6.3037	6.3102	6.3153	6.3192	6.3222	6.3246
	2	5.0095	5.1861	5.3123	5.4037	5.4701	5.5184	5.5534	5.5788	5.5970	5.6101	5.6193	5.6257	5.6302	5.6333	5.6354	5.6368	5.6377	5.6383	5.6387	5.6390	5.6391
	3	4.8006	4.9437	5.0402	5.1056	5.1498	5.1795	5.1993	5.2124	5.2208	5.2263	5.2297	5.2318	5.2331	5.2339	5.2343	5.2346	5.2347	5.2348	5.2349		5.2349
	4	4.6301	4.7481	4.8235	4.8714	4.9017	4.9205	4.9320	4.9390	4.9430	4.9454	4.9467	4.9474	4.9478	4.9480	4.9481						4.9481
	5	4.4844	4.5830	4.6426	4.6783	4.6993	4.7114	4.7183	4.7220	4.7241	4.7251	4.7256	4.7259	4.7260	4.7260							4.7260
	6	4.3566	4.4396	4.4872	4.5140	4.5288	4.5367	4.5409	4.5429	4.5439	4.5444	4.5446	4.5447									4.5447
	7	4.2425	4.3128	4.3510	4.3714	4.3819	4.3871	4.3896	4.3908	4.3913	4.3915	4.3916										4.3916
	8	4.1393	4.1991	4.2300	4.2455	4.2530	4.2565	4.2580	4.2587	4.2589	4.2590	4.2591										4.2591
	9	4.0450	4.0961	4.1212	4.1331	4.1385	4.1408	4.1417	4.1421	4.1422	4.1423											4.1423
10^{-2}	1	3.9582	4.0020	4.0224	4.0316	4.0355	4.0370	4.0376	4.0378													4.0378
	2	3.3403	3.3507	3.3537	3.3545	3.3547																3.3547
	3	2.9558	2.9585	2.9590																		2.9590
	4	2.6804	2.6812	2.6812																		2.6812
	5	2.4677	2.4679																			2.4679
	6	2.2952																				2.2952

Values of $M(u,\beta)$ are equal to $W(u)$ for u greater than $6\cdot10^{-2}$ and all values of β

u	β	50	52	54	56	58	60	62	64	66	68	70	72	74	76	78	80	82	84	86	88	90
	0	9.2102	9.2886	9.3641	9.4368	9.5069	9.5747	9.6403	9.7037	9.7653	9.8249	9.8829	9.9392	9.9940	10.0473	10.0992	10.1498	10.1992	10.2474	10.2944	10.3404	10.3853
10^{-6}	1	9.0977	9.1716	9.2426	9.3108	9.3765	9.4398	9.5008	9.5598	9.6168	9.6720	9.7255	9.7773	9.8276	9.8764	9.9239	9.9700	10.0148	10.0585	10.1011	10.1425	10.1830
	2	9.0510	9.1231	9.1922	9.2585	9.3223	9.3838	9.4430	9.5001	9.5553	9.6086	9.6602	9.7102	9.7586	9.8056	9.8512	9.8955	9.9385	9.9803	10.0210	10.0606	10.0992
	3	9.0152	9.0859	9.1535	9.2185	9.2809	9.3409	9.3986	9.4543	9.5081	9.5600	9.6102	9.6588	9.7058	9.7513	9.7955	9.8384	9.8800	9.9204	9.9597	9.9979	10.0351
	4	8.9851	9.0545	9.1210	9.1847	9.2459	9.3047	9.3613	9.4158	9.4684	9.5191	9.5681	9.6155	9.6613	9.7057	9.7487	9.7904	9.8308	9.8700	9.9081	9.9452	9.9812
	5	8.9586	9.0269	9.0924	9.1550	9.2152	9.2729	9.3285	9.3819	9.4335	9.4831	9.5311	9.5774	9.6222	9.6655	9.7075	9.7481	9.7875	9.8257	9.8628	9.8988	9.9338
	6	8.9346	9.0020	9.0665	9.1282	9.1874	9.2442	9.2988	9.3513	9.4019	9.4507	9.4977	9.5431	9.5869	9.6293	9.6703	9.7101	9.7485	9.7858	9.8220	9.8571	9.8911
	7	8.9126	8.9791	9.0427	9.1036	9.1619	9.2179	9.2716	9.3232	9.3730	9.4208	9.4670	9.5115	9.5545	9.5961	9.6362	9.6751	9.7127	9.7492	9.7845	9.8187	9.8519
	8	8.8921	8.9578	9.0206	9.0807	9.1382	9.1933	9.2463	9.2971	9.3460	9.3931	9.4385	9.4822	9.5244	9.5652	9.6046	9.6426	9.6795	9.7151	9.7497	9.7831	9.8156
	9	8.8729	8.9378	9.0000	9.0592	9.1160	9.1703	9.2225	9.2726	9.3208	9.3671	9.4118	9.4548	9.4962	9.5362	9.5749	9.6122	9.6483	9.6833	9.7171	9.7498	9.7815
10^{-5}	1	8.8547	8.9190	8.9803	9.0389	9.0949	9.1486	9.2001	9.2495	9.2970	9.3426	9.3865	9.4288	9.4696	9.5089	9.5469	9.5835	9.6189	9.6532	9.6863	9.7183	9.7494
	2	8.7087	8.7673	8.8229	8.8759	8.9263	8.9743	9.0202	9.0640	9.1059	9.1461	9.1845	9.2213	9.2566	9.2905	9.3230	9.3542	9.3843	9.4132	9.4410	9.4677	9.4935
	3	8.5976	8.6519	8.7033	8.7521	8.7983	8.8422	8.8839	8.9237	8.9615	8.9975	9.0319	9.0647	9.0960	9.1260	9.1546	9.1819	9.2081	9.2332	9.2572	9.2802	9.3023
	4	8.5047	8.5555	8.6035	8.6488	8.6916	8.7321	8.7705	8.8069	8.8414	8.8742	8.9053	8.9349	8.9630	8.9898	9.0153	9.0396	9.0628	9.0848	9.1059	9.1260	9.1451
	5	8.4235	8.4713	8.5163	8.5587	8.5986	8.6362	8.6717	8.7053	8.7370	8.7671	8.7955	8.8224	8.8479	8.8721	8.8950	8.9167	8.9374	8.9570	8.9756	8.9932	9.0100
	6	8.3507	8.3959	8.4383	8.4780	8.5154	8.5505	8.5836	8.6147	8.6440	8.6716	8.6977	8.7223	8.7455	8.7675	8.7882	8.8078	8.8263	8.8438	8.8603	8.8760	8.8908
	7	8.2843	8.3271	8.3672	8.4046	8.4397	8.4726	8.5034	8.5324	8.5596	8.5851	8.6091	8.6317	8.6530	8.6730	8.6918	8.7095	8.7262	8.7419	8.7567	8.7706	8.7838
	8	8.2229	8.2636	8.3016	8.3370	8.3700	8.4009	8.4297	8.4568	8.4821	8.5057	8.5279	8.5487	8.5682	8.5865	8.6036	8.6197	8.6348	8.6490	8.6623	8.6747	8.6864
	9	8.1658	8.2045	8.2405	8.2740	8.3052	8.3343	8.3614	8.3866	8.4102	8.4322	8.4528	8.4720	8.4899	8.5067	8.5223	8.5370	8.5507	8.5635	8.5754	8.5866	8.5970
10^{-4}	1	8.1122	8.1491	8.1833	8.2151	8.2446	8.2720	8.2974	8.3211	8.3431	8.3636	8.3827	8.4005	8.4170	8.4324	8.4468	8.4601	8.4726	8.4842	8.4949	8.5050	8.5143
	2	7.6962	7.7203	7.7421	7.7618	7.7797	7.7958	7.8104	7.8236	7.8355	7.8463	7.8560	7.8648	7.8727	7.8798	7.8862	7.8920	7.8972	7.9019	7.9061	7.9098	7.9132
	3	7.3997	7.4163	7.4309	7.4439	7.4553	7.4654	7.4742	7.4820	7.4889	7.4949	7.5002	7.5048	7.5088	7.5124	7.5154	7.5181	7.5204	7.5225	7.5242	7.5258	7.5271
	4	7.1663	7.1780	7.1881	7.1968	7.2043	7.2108	7.2163	7.2211	7.2251	7.2286	7.2315	7.2341	7.2362	7.2380	7.2395	7.2408	7.2419	7.2428	7.2436	7.2442	7.2447
	5	6.9734	6.9818	6.9888	6.9948	6.9998	7.0040	7.0075	7.0105	7.0129	7.0150	7.0167	7.0181	7.0192	7.0202	7.0209	7.0216	7.0221	7.0225	7.0228	7.0231	7.0233
	6	6.8090	6.8151	6.8201	6.8242	6.8276	6.8304	6.8327	6.8345	6.8360	6.8372	6.8382	6.8390	6.8396	6.8401	6.8405	6.8408	6.8411	6.8413	6.8414	6.8416	6.8417
	7	6.6661	6.6705	6.6741	6.6770	6.6793	6.6811	6.6826	6.6838	6.6847	6.6854	6.6860	6.6864	6.6868	6.6871	6.6873	6.6874	6.6875	6.6876	6.6877	6.6878	6.6878
	8	6.5397	6.5430	6.5456	6.5476	6.5492	6.5504	6.5514	6.5521	6.5527	6.5531	6.5535	6.5537	6.5539	6.5541	6.5542	6.5543	6.5543	6.5544	6.5544	6.5544	6.5544
	9	6.4267	6.4291	6.4310	6.4324	6.4335	6.4343	6.4350	6.4354	6.4358	6.4361	6.4363	6.4364	6.4365	6.4366	6.4366	6.4367	6.4367	6.4367	6.4368		6.4368
10^{-3}	1	6.3246	6.3263	6.3277	6.3287	6.3294	6.3300	6.3304	6.3307	6.3310	6.3311	6.3312	6.3313	6.3314	6.3314	6.3315						6.3315
	2	5.6391	5.6392	5.6393	5.6393	5.6393																5.6393

Values of $M(u,\beta)$ are equal to $W(u)$ for u greater than $2\cdot10^{-3}$ and all values of β

u	β	90	92	94	96	98	100	120	140	160	180	200	220	240	260	280	300	320	340	360	380	400
	0	10.3853	10.4292	10.4722	10.5143	10.5555	10.5959	10.9604	11.2686	11.5355	11.7709	11.9815	12.1720	12.3459	12.5058	12.6539	12.7918	12.9207	13.0418	13.1560	13.2640	13.366
10^{-6}	1	10.1830	10.2224	10.2609	10.2986	10.3353	10.3712	10.6909	10.9543	11.1765	11.3674	11.5335	11.6797	11.8093	11.9252	12.0294	12.1234	12.2088	12.2865	12.3574	12.4224	12.482
	2	10.0992	10.1368	10.1735	10.2092	10.2441	10.2782	10.5795	10.8247	11.0288	11.2018	11.3502	11.4788	11.5912	11.6900	11.7773	11.8549	11.9240	11.9858	12.0411	12.0908	12.135
	3	10.0351	10.0713	10.1065	10.1409	10.1744	10.2070	10.4944	10.7258	10.9165	11.0760	11.2113	11.3271	11.4269	11.5134	11.5888	11.6547	11.7125	11.7633	11.8081	11.8475	11.882
	4	9.9812	10.0162	10.0503	10.0834	10.1158	10.1473	10.4231	10.6431	10.8225	10.9712	11.0958	11.2012	11.2909	11.3677	11.4337	11.4906	11.5398	11.5823	11.6192	11.6511	11.678
	5	9.9338	9.9678	10.0008	10.0330	10.0643	10.0948	10.3605	10.5706	10.7404	10.8797	10.9953	11.0919	11.1733	11.2420	11.3003	11.3499	11.3922	11.4282	11.4589	11.4850	11.507
	6	9.8911	9.9242	9.9563	9.9876	10.0179	10.0475	10.3042	10.5055	10.6668	10.7979	10.9056	10.9945	11.0688	11.1307	11.1826	11.2261	11.2626	11.2933	11.3191	11.3408	11.358
	7	9.8519	9.8842	9.9155	9.9459	9.9754	10.0041	10.2526	10.4461	10.5997	10.7234	10.8241	10.9065	10.9744	11.0304	11.0767	11.1151	11.1469	11.1732	11.1950	11.2130	11.227
	8	9.8156	9.8470	9.8776	9.9072	9.9360	9.9640	10.2049	10.3911	10.5377	10.6548	10.7492	10.8257	10.8880	10.9388	10.9804	11.0144	11.0422	11.0649	11.0834	11.0984	11.110
	9	9.7815	9.8123	9.8421	9.8710	9.8991	9.9263	10.1602	10.3397	10.4800	10.5910	10.6797	10.7509	10.8082	10.8545	10.8919	10.9221	10.9465	10.9661	10.9819	10.9945	11.004
10^{-5}	1	9.7494	9.7794	9.8086	9.8368	9.8642	9.8908	10.1182	10.2914	10.4258	10.5313	10.6147	10.6811	10.7340	10.7763	10.8100	10.8369	10.8584	10.8754	10.8888	10.8994	10.907
	2	9.4935	9.5184	9.5423	9.5654	9.5876	9.6091	9.7870	9.9141	10.0059	10.0726	10.1211	10.1563	10.1818	10.2001	10.2131	10.2224	10.2288	10.2333	10.2364	10.2385	10.239
	3	9.3023	9.3234	9.3437	9.3631	9.3818	9.3997	9.5441	9.6413	9.7071	9.7516	9.7815	9.8014	9.8145	9.8230	9.8284	9.8319	9.8340	9.8353	9.8361	9.8365	9.836
	4	9.1451	9.1634	9.1808	9.1975	9.2134	9.2285	9.3478	9.4238	9.4721	9.5026	9.5216	9.5332	9.5401	9.5442	9.5466	9.5479	9.5486	9.5490	9.5492	9.5493	9.549
	5	9.0100	9.0260	9.0411	9.0555	9.0692	9.0822	9.1819	9.2421	9.2781	9.2993	9.3116	9.3185	9.3222	9.3243	9.3253	9.3258	9.3261	9.3262	9.3263		9.326
	6	8.8908	8.9048	8.9181	8.9306	8.9425	8.9538	9.0378	9.0859	9.1130	9.1279	9.1359	9.1401	9.1422	9.1432	9.1436	9.1438	9.1439	9.1440	9.1440		9.144
	7	8.7838	8.7962	8.8078	8.8188	8.8292	8.8389	8.9101	8.9489	8.9694	8.9800	8.9853	8.9878	8.9890	8.9895	8.9897	8.9897	8.9898	8.9898			8.989
	8	8.6864	8.6974	8.7077	8.7174	8.7264	8.7349	8.7956	8.8269	8.8426	8.8502	8.8537	8.8553	8.8559	8.8562	8.8563	8.8563					8.856
	9	8.5970	8.6068	8.6159	8.6244	8.6324	8.6399	8.6917	8.7172	8.7293	8.7347	8.7370	8.7380	8.7384	8.7385	8.7385	8.7386					8.738
10^{-4}	1	8.5143	8.5231	8.5312	8.5387	8.5457	8.5522	8.5967	8.6175	8.6268	8.6307	8.6323	8.6329	8.6331								8.633
	2	7.9132	7.9162	7.9189	7.9213	7.9235	7.9254	7.9361	7.9392	7.9399	7.9401	7.9402										7.940
	3	7.5271	7.5282	7.5292	7.5300	7.5307	7.5314	7.5342	7.5347													7.534
	4	7.2447	7.2452	7.2456	7.2459	7.2461	7.2463	7.2471	7.2472													7.247
	5	7.0233	7.0235	7.0237	7.0238	7.0239	7.0239	7.0242														7.024
	6	6.8417	6.8417	6.8418	6.8418	6.8419	6.8419	6.8419														6.842
	7	6.6878	6.6878	6.6878	6.6879																	6.687
	8	6.5544	6.5544	6.5544																		6.554
	9																					

Values of $M(u,\beta)$ are equal to $W(u)$ for u greater than $8\cdot10^{-4}$ and all values of β

(11) $L_1^*(u, \omega)$ is defined by

$$L_1^*(u,\omega) = e^{-\omega} \int_1^{\infty} \exp\left(-uy + \frac{\omega}{y}\right) \frac{dy}{y}$$

and can be easily tabulated for a practical range of u and ω, using a tabulation of a related function [61].

(12) $M(u,\beta)$ defined [21] by

$$M(u,\beta) = -M(u, -\beta) = \int_u^{\infty} (1/y) \exp(-y) \operatorname{erf}(\beta\sqrt{y})\, dy$$

and is extensively tabulated [22]. Sufficient values for practical use are given in Table IV. This function may be approximated as follows:

for	$u < 0.05/\beta^2 < 0.01,$	$M(u,\beta) \approx 2[\sinh^{-1}\beta - 2\beta\sqrt{u/\pi}]$
	$u < 0.05/\beta^2,$	$M(u,\beta) \approx 2[\sinh^{-1}\beta - \beta \operatorname{erf}(\sqrt{u})]$
	$u > 5/\beta^2,$	$M(u,\beta) \approx W(u)$

(13) $S(\tau,\rho)$ is defined by

$$S(\tau,\rho) = (4/\pi) \int_0^{\infty} [1 - \exp(-\tau u^2)]R\, du$$

where

$$R = [J_1(u)Y_0(\rho u) - Y_1(u)J_0(\rho u)]/u^2[J_1^2(u) + Y_1^2(u)]$$

a few values of which are given in Table VI. The table is adapted from that of a function [23] that is equal to $(1/2\pi)S(\tau,\rho)$. The function may be approximated [23] as follows:

$$S(\tau,\rho) \approx W(\rho^2/4\tau), \qquad \text{if} \qquad \tau > 20$$

(14) $\sinh^{-1}x$ is the inverse hyperbolic sine, defined by

$$\sinh^{-1}(x) = -\sinh^{-1}(-x) = \ln(x + \sqrt{x^2 + 1})$$

tabular values of which are available [15]. It may be approximated by

$$\sinh^{-1}x \approx x, \qquad \text{if} \qquad x < 0.1$$
$$\sinh^{-1}x \approx \ln 2x, \qquad \text{if} \qquad x > 10$$

(15) $V(\tau,\rho)$ may be called the *gravity well function* and is defined [24] by

$$V(\tau,\rho) = \int_0^{\infty} (1/u)J_0(u\rho)\{1 - \exp(-\tau u \tanh u)\}\, du$$

It is available in tabular form [25]. Table VII is constructed by graphical interpolation from available tabulation [25]; thus, it may be in error in the

TABLE V. Values of the Function $L(\mu, O)$

u	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
0	0.0	0.3422	0.5471	0.7020	0.8256	0.9271	1.0119	1.0836	1.1447	1.1972	1.2425
1	1.2425	1.2818	1.3159	1.3457	1.3718	1.3946	1.4146	1.4323	1.4478	1.4616	1.4737
2	1.4737	1.4844	1.4939	1.5023	1.5098	1.5164	1.5223		1.5321		1.5400
3	1.5400		1.5462		1.5511		1.5550		1.5582		1.5606
4	1.5606		1.5626		1.5642		1.5655		1.5666		1.5674
5	1.5674		1.5681		1.5686		1.5690		1.5694		1.5696
6	1.5696		1.5699		1.5700		1.5702		1.5703		1.5704
7	1.5704		1.5705		1.5705		1.5706		1.5706		1.5707
8	1.57066		1.57068		1.57071		1.57072		1.57074		1.57075
9	1.57075		1.57076		1.57076		1.57077		1.57078		1.57078
10	1.57078		1.57078		1.57078		1.57078		1.57079		1.57079

TABLE VI. Values of the Function $S(\tau, \rho)$

τ \ ρ	1	2	5	10
0.10	0.616			
0.20	0.842			
0.30	1.005			
0.40	1.131			
0.50	1.244			
0.60	1.344			
0.70	1.420			
0.80	1.483			
1.00	1.608	0.440	0.0126	0.000
1.20	1.722			
1.5	1.860			
2.0	2.048			
2.5	2.199			
3.0	2.337			
4.0	2.551			
5	2.727	1.407	0.192	0.0012
6	2.865			
8	3.104			
10	3.305	1.948	0.488	0.0302
12	3.456	2.098	0.591	0.0528
15	3.657	2.287	0.729	0.0905
20	3.921	2.551	0.925	0.162
25	4.122	2.752	1.088	0.236
30	4.298	2.915	1.230	0.309
50	4.775	3.405	1.658	0.578
100	5.441	4.059	2.274	1.058
500	7.037	5.642	3.820	2.476
1,000	7.716	6.333	4.511	3.142
5,000	9.324	7.942	6.107	4.725
10,000	10.015	8.633	6.798	5.416
25,000	10.933	9.550	7.716	6.333

last shown decimal. The function may be approximated, for practical purposes, by

a. for $\tau < 0.05$,

$$V(\tau,\rho) \approx \sinh^{-1}(1/\rho) + \sinh^{-1}(\tau/\rho) - \sinh^{-1}[(1+\tau)/\rho]$$

b. for $\tau < 0.01$,

$$V(\tau,\rho) \approx \sinh^{-1}(\tau/\rho) - \tau/\sqrt{1+\rho^2}$$

c. for $\tau < 0.01$ and $\tau/\rho > 10$,

$$V(\tau,\rho) \approx \ln(2\tau/\rho)$$

d. for $\tau > 5$,

$$V(\tau,\rho) \approx 0.5W(\rho^2/4\tau)$$

TABLE VII. Values of the Function $V(\tau, \rho)$

	τ \ ρ	10^{-3}: 1	2	3	4	5	6	7	8	9	10^{-2}: 1	2	3	4	5	6	7	8	9	10
10^{-2}	1	2.99	2.30	1.90	1.64	1.42	1.28	1.15	1.04	0.950	0.875	0.474	0.322	0.240	0.192	0.158	0.135	0.118	0.104	0.093
	2	3.68	2.97	2.58	2.30	2.09	1.92	1.76	1.64	1.52	1.42	0.860	0.610	0.468	0.378	0.316	0.270	0.236	0.210	0.187
	3	4.08	3.40	3.00	2.70	2.46	2.28	2.13	2.00	1.88	1.79	1.18	0.860	0.675	0.555	0.465	0.400	0.350	0.310	0.278
	4	4.35	3.68	3.26	2.98	2.75	2.58	2.42	2.29	2.17	2.06	1.42	1.07	0.850	0.710	0.600	0.525	0.460	0.410	0.368
	5	4.58	3.90	3.49	3.20	2.96	2.79	2.64	2.50	2.38	2.28	1.60	1.24	1.010	0.850	0.725	0.630	0.560	0.500	0.450
	6	4.76	4.06	3.65	3.36	3.15	2.96	2.80	2.68	2.56	2.45	1.78	1.40	1.15	0.970	0.840	0.735	0.650	0.585	0.530
	7	4.92	4.20	3.80	3.51	3.30	3.12	2.96	2.82	2.70	2.60	1.91	1.54	1.28	1.09	0.950	0.835	0.740	0.670	0.610
	8	5.08	4.34	3.94	3.65	3.42	3.24	3.09	2.95	2.84	2.72	2.04	1.65	1.39	1.20	1.04	0.925	0.825	0.750	0.680
	9	5.18	4.47	4.05	3.75	3.54	3.35	3.20	3.05	2.95	2.84	2.14	1.75	1.50	1.29	1.14	1.02	0.910	0.825	0.750
10^{-1}	1	5.24	4.54	4.14	3.85	3.63	3.45	3.30	3.15	3.04	2.94	2.25	1.85	1.58	1.38	1.22	1.09	0.985	0.890	0.815
	2	5.85	5.15	4.78	4.50	4.28	4.10	3.93	3.80	3.66	3.56	2.87	2.46	2.20	1.98	1.80	1.65	1.52	1.42	1.32
	3	6.24	5.50	5.12	4.85	4.61	4.43	4.28	4.14	4.01	3.90	3.24	2.84	2.54	2.32	2.14	1.98	1.85	1.74	1.64
	4	6.45	5.75	5.35	5.08	4.85	4.67	4.50	4.38	4.26	4.15	3.46	3.05	2.76	2.54	2.36	2.20	2.07	1.96	1.86
	5	6.65	6.00	5.58	5.25	5.00	4.85	4.70	4.55	4.45	4.30	3.65	3.24	2.95	2.72	2.52	2.38	2.24	2.14	2.03
	6	6.75	6.10	5.65	5.40	5.15	4.98	4.82	4.68	4.56	4.45	3.76	3.37	3.09	2.85	2.67	2.50	2.38	2.26	2.16
	7	6.88	6.20	5.80	5.50	5.25	5.08	4.92	4.80	4.68	4.55	3.90	3.50	3.20	2.99	2.80	2.64	2.50	2.38	2.28
	8	7.00	6.25	5.85	5.60	5.35	5.20	5.00	4.90	4.80	4.65	3.96	3.55	3.26	3.05	2.86	2.71	2.58	2.46	2.36
	9	7.10	6.35	6.00	5.70	5.50	5.30	5.12	5.00	4.90	4.75	4.05	3.65	3.36	3.15	2.96	2.80	2.66	2.55	2.45
1	1	7.14	6.45	6.05	5.75	5.55	5.35	5.20	5.05	4.95	4.83	4.10	3.74	3.45	3.22	3.04	2.90	2.75	2.64	2.54
	2	7.60	6.88	6.45	6.15	5.92	5.75	5.60	5.50	5.35	5.25	4.59	4.18	3.90	3.68	3.50	3.34	3.20	3.09	2.97
	3	7.85	7.15	6.70	6.45	6.20	6.00	5.85	5.75	5.60	5.50	4.82	4.42	4.12	3.90	3.72	3.57	3.45	3.31	3.20
	4	8.00	7.28	6.85	6.58	6.35	6.15	6.00	5.90	5.75	5.70	4.95	4.55	4.26	4.04	3.86	3.70	3.59	3.46	3.36
	5	8.15	7.35	7.00	6.65	6.50	6.25	6.10	6.00	5.85	5.80	5.05	4.68	4.40	4.19	4.00	3.85	3.71	3.60	3.49
	6	8.20	7.50	7.10	6.75	6.55	6.35	6.20	6.10	5.95	5.85	5.20	4.78	4.50	4.26	4.09	3.92	3.80	3.69	3.59
	7	8.25	7.55	7.15	6.85	6.62	6.40	6.30	6.20	6.05	5.95	5.25	4.85	[illegible]	4.35	4.18	4.00	3.90	3.78	3.66
	8	8.30	7.60	7.20	6.90	6.70	6.50	6.35	6.25	6.10	6.05	5.30	4.92	[illegible]	4.40	4.25	4.10	3.95	3.82	3.74
	9	8.32	7.65	7.25	7.00	6.75	6.55	6.40	6.30	6.15	6.10	5.35	5.00	4.70	4.49	4.30	4.15	4.00	3.90	3.80
	10	8.35	7.75	7.35	7.05	6.80	6.60	6.45	6.35	6.20	6.14	5.40	5.02	4.80	4.52	4.35	4.19	4.05	3.92	3.84

	τ \ ρ	10^{-1}: 1	2	3	4	5	6	7	8	9	1: 1	2	3	4	5
10^{-2}	1	0.093	0.0430	0.0264	0.0180	0.0132	0.0100	0.0078	0.0062	0.0049	0.0040	0.00057	0.00015		
	2	0.187	0.0865	0.0530	0.0365	0.0268	0.0205	0.0160	0.0125	0.0100	0.0081	0.00118	0.00020		
	3	0.278	0.130	0.0800	0.0550	0.0405	0.0310	0.0240	0.0190	0.0150	0.0122	0.00184	0.00032		
	4	0.368	0.174	0.107	0.0735	0.0540	0.0415	0.0322	0.0255	0.0202	0.0165	0.00244	0.00043		
	5	0.450	0.215	0.133	0.0920	0.0675	0.0520	0.0400	0.0320	0.0255	0.0206	0.00305	0.00055		
	6	0.530	0.257	0.160	0.110	0.0810	0.0610	0.0478	0.0380	0.0305	0.0250	0.00365	0.00065		
	7	0.610	0.298	0.186	0.130	0.0950	0.0725	0.0565	0.0450	0.0360	0.0292	0.00430	0.00078		
	8	0.680	0.340	0.214	0.148	0.108	0.0825	0.0645	0.0510	0.0412	0.0336	0.00500	0.00090		
	9	0.750	0.378	0.236	0.164	0.122	0.0930	0.0730	0.0585	0.0470	0.0380	0.00570	0.00105		
10^{-1}	1	0.815	0.415	0.260	0.180	0.134	0.103	0.0805	0.0640	0.0515	0.0420	0.00635	0.00118		
	2	1.32	0.750	0.500	0.359	0.268	0.208	0.165	0.132	0.107	0.0880	0.0145	0.00278		
	3	1.64	1.02	0.700	0.515	0.392	0.308	0.246	0.200	0.164	0.135	0.0238	0.00490		
	4	1.86	1.22	0.870	0.650	0.510	0.405	0.328	0.268	0.220	0.182	0.0350	0.00750	0.00160	0.00038
	5	2.03	1.37	1.00	0.770	0.610	0.490	0.400	0.330	0.275	0.230	0.0450	0.0104	0.00240	0.00056
	6	2.16	1.49	1.12	0.875	0.700	0.570	0.468	0.390	0.325	0.276	0.0580	0.0138	0.00320	0.00080
	7	2.28	1.60	1.22	0.965	0.775	0.640	0.525	0.445	0.375	0.320	0.0715	0.0175	0.00425	0.00108
	8	2.36	1.69	1.30	1.04	0.850	0.715	0.600	0.500	0.425	0.364	0.084C	0.0212	0.00525	0.00140
	9	2.45	1.75	1.38	1.11	0.920	0.775	0.650	0.550	0.475	0.404	0.0980	0.0260	0.00630	0.00165
1	1	2.54	1.85	1.45	1.18	0.975	0.825	0.700	0.595	0.510	0.444	0.113	0.0310	0.00840	0.00235
	2	2.97	2.29	1.88	1.60	1.38	1.22	1.07	0.950	0.840	0.750	0.259	0.0950	0.0330	0.0115
	3	3.20	2.50	2.10	1.82	1.60	1.42	1.28	1.15	1.05	0.960	0.388	0.165	0.0700	0.0275
	4	3.36	2.66	2.25	1.97	1.75	1.58	1.42	1.30	1.20	1.10	0.495	0.235	0.112	0.0535
	5	3.49	2.78	2.38	2.09	1.87	1.69	1.54	1.42	1.30	1.21	0.580	0.300	0.150	0.0715
	6	3.59	2.90	2.47	2.18	1.95	1.78	1.65	1.52	1.40	1.30	0.660	0.360	0.195	0.0990
	7	3.66	2.96	2.55	2.25	2.04	1.85	1.70	1.58	1.48	1.38	0.730	0.415	0.230	0.125
	8	3.74	3.00	2.60	2.32	2.11	1.94	1.79	1.66	1.55	1.44	0.790	0.465	0.272	0.155
	9	3.80	3.09	2.67	2.39	2.17	2.00	1.85	1.72	1.60	1.50	0.850	0.515	0.307	0.182
	10	3.84	3.12	2.74	2.45	2.24	2.05	1.90	1.77	1.65	1.55	0.890	0.550	0.340	0.210

For $\tau > 5$, $V(\tau,\rho) \approx 0.5\,W(\rho^2/4\tau)$

For $\tau < 0.01$, $V(\tau,\rho) \approx \sinh^{-1}(\tau/\rho) - \frac{\tau}{\sqrt{1+\rho^2}}$

For $\tau < 0.01$ & $\tau/\rho > 10$, $V(\tau,\rho) \approx \sinh^{-1}(1/\rho) - \ln\left(\frac{\tau+1}{\tau}\right)$

(16) $W(u)$, known in the field of hydrology as the *well function for nonleaky aquifers*, is equal to the exponential integral; namely,

$$W(u) = -E_i(-u) = \int_u^\infty (1/y)\exp(-y)\,dy$$

The function is extensively tabulated [26, 27]. Sufficient values of this function are given in Table VIII for $W(u,0) = W(u)$. Its series expansion is

$$W(u) = -0.5772 - \ln u - \sum_{n=1}^{\infty} (-1)^n u^n / n \cdot n\,!$$

and for $u < 0.05$, it is approximated by

$$W(u) = -0.5772 - \ln u = \ln(0.562/u)$$

(17) $W(u,\beta)$, known as the *well function for leaky aquifers*, is defined by

$$W(u,\beta) = \int_u^\infty (1/y)\exp(-y - \beta^2/4y)\,dy$$

and is extensively tabulated [28]. It is given in Table VIII for a sufficient practical range of u and β. The following relations may be useful in practical computations:

$$W(0,\beta) = 2K_0(\beta), \qquad W(u,0) = W(u),$$
$$W(u,\beta) = 2K_0(\beta) - W(\beta^2/4u,\beta)$$

and

$$W(u,\beta) \approx W(u) \quad \text{for} \quad u > 2\beta$$
$$\approx W(u) \quad \text{for} \quad u > 5\beta^2, \quad \text{if} \quad \beta < 0.1$$
$$W(u,\beta) \approx 2K_0(\beta) - I_0(\beta)W(\beta^2/4u) \quad \text{for} \quad u < \beta^2/20, \quad \text{if} \quad u < 1$$

(18) $Y_0(z)$ and $Y_1(z)$ are the zero- and first-order Bessel functions of the second kind [12]; tabular values are available [15]. They may be approximated by

a. for $z < 0.01$,

$$Y_0(z) \approx (2/\pi)[0.5772 + \ln(z/2)],$$
$$Y_1(z) \approx -2/\pi z$$

b. for $z > 16$,

$$Y_0(z) \approx \sqrt{2/\pi z}\,\sin(z - \pi/4),$$
$$Y_1(z) \approx -\sqrt{2/\pi z}\,\cos(z - \pi/4)$$

TABLE VIII. Values of the Function $W(\mu, \beta)$

u \ β	0	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.01
0	∞	14.0474	12.6611	11.8502	11.2748	10.8286	10.4640	10.1557	9.8887	9.6532	9.4425
.000001	13.2383	13.0031	12.4417	11.8153	11.2711	10.8283	10.4640	10.1557	9.8887		
.000002	12.5451	12.4240	12.1013	11.6716	11.2259	10.8174	10.4619	10.1554	9.8886	9.6532	
.000003	12.1397	12.0581	11.8322	11.5098	11.1462	10.7849	10.4509	10.1523	9.8879	9.6530	9.4425
.000004	11.8520	11.7905	11.6168	11.3597	11.0555	10.7374	10.4291	10.1436	9.8849	9.6521	9.4422
.000005	11.6289	11.5795	11.4384	11.2248	10.9642	10.6822	10.3993	10.1290	9.8786	9.6496	9.4413
.000006	11.4465	11.4053	11.2866	11.1040	10.8764	10.6240	10.3640	10.1094	9.8686	9.6450	9.4394
.000007	11.2924	11.2570	11.1545	10.9951	10.7933	10.5652	10.3255	10.0862	9.8555	9.6382	9.4361
.000008	11.1589	11.1279	11.0377	10.8962	10.7151	10.5072	10.2854	10.0602	9.8398	9.6292	9.4313
.000009	11.0411	11.0135	10.9330	10.8059	10.6416	10.4508	10.2446	10.0324	9.8219	9.6182	9.4251
.00001	10.9357	10.9109	10.8382	10.7228	10.5725	10.3963	10.2038	10.0034	9.8024	9.6059	9.4176
.00002	10.2426	10.2301	10.1932	10.1332	10.0522	9.9530	9.8386	9.7126	9.5781	9.4383	9.2961
.00003	9.8371	9.8288	9.8041	9.7635	9.7081	9.6392	9.5583	9.4671	9.3674	9.2611	9.1499
.00004	9.5495	9.5432	9.5246	9.4940	9.4520	9.3992	9.3366	9.2653	9.1863	9.1009	9.0102
.00005	9.3263	9.3213	9.3064	9.2818	9.2480	9.2052	9.1542	9.0957	9.0304	8.9591	8.8827
.00006	9.1440	9.1398	9.1274	9.1069	9.0785	9.0426	8.9996	8.9500	8.8943	8.8332	8.7673
.00007	8.9899	8.9863	8.9756	8.9580	8.9336	8.9027	8.8654	8.8224	8.7739	8.7204	8.6625
.00008	8.8563	8.8532	8.8439	8.8284	8.8070	8.7798	8.7470	8.7090	8.6661	8.6186	8.5669
.00009	8.7386	8.7358	8.7275	8.7138	8.6947	8.6703	8.6411	8.6071	8.5686	8.5258	8.4792
.0001	8.6332	8.6308	8.6233	8.6109	8.5937	8.5717	8.5453	8.5145	8.4796	8.4407	8.3983
.0002	7.9402	7.9390	7.9352	7.9290	7.9203	7.9092	7.8958	7.8800	7.8619	7.8416	7.8192
.0003	7.5348	7.5340	7.5315	7.5274	7.5216	7.5141	7.5051	7.4945	7.4823	7.4686	7.4534
.0004	7.2472	7.2466	7.2447	7.2416	7.2373	7.2317	7.2249	7.2169	7.2078	7.1974	7.1859
.0005	7.0242	7.0237	7.0222	7.0197	7.0163	7.0118	7.0063	6.9999	6.9926	6.9843	6.9750
.0006	6.8420	6.8416	6.8403	6.8383	6.8353	6.8316	6.8271	6.8218	6.8156	6.8086	6.8009
.0007	6.6879	6.6876	6.6865	6.6848	6.6823	6.6790	6.6752	6.6706	6.6653	6.6594	6.6527
.0008	6.5545	6.5542	6.5532	6.5517	6.5495	6.5467	6.5433	6.5393	6.5347	6.5295	6.5237
.0009	6.4368	6.4365	6.4357	6.4344	6.4324	6.4299	6.4269	6.4233	6.4192	6.4146	6.4094
.001	6.3315	6.3313	6.3305	6.3293	6.3276	6.3253	6.3226	6.3194	6.3157	6.3115	6.3069
.002	5.6394	5.6393	5.6389	5.6383	5.6374	5.6363	5.6350	5.6334	5.6315	5.6294	5.6271
.003	5.2349	5.2348	5.2346	5.2342	5.2336	5.2329	5.2320	5.2310	5.2297	5.2283	5.2267
.004	4.9482	4.9482	4.9480	4.9477	4.9472	4.9467	4.9460	4.9453	4.9443	4.9433	4.9421
.005	4.7261	4.7260	4.7259	4.7256	4.7253	4.7249	4.7244	4.7237	4.7230	4.7222	4.7212
.006	4.5448	4.5448	4.5447	4.5444	4.5441	4.5438	4.5433	4.5428	4.5422	4.5415	4.5407
.007	4.3916	4.3916	4.3915	4.3913	4.3910	4.3908	4.3904	4.3899	4.3894	4.3888	4.3882
.008	4.2591	4.2590	4.2590	4.2588	4.2586	4.2583	4.2580	4.2576	4.2572	4.2567	4.2561
.009	4.1423	4.1423	4.1422	4.1420	4.1418	4.1416	4.1413	4.1410	4.1406	4.1401	4.1396
.01	4.0379	4.0379	4.0378	4.0377	4.0375	4.0373	4.0371	4.0368	4.0364	4.0360	4.0356
.02	3.3547	3.3547	3.3547	3.3546	3.3545	3.3544	3.3543	3.3542	3.3540	3.3538	3.3536
.03	2.9591	2.9591	2.9591	2.9590	2.9590	2.9589	2.9589	2.9588	2.9587	2.9585	2.9584
.04	2.6813	2.6812	2.6812	2.6812	2.6812	2.6811	2.6810	2.6810	2.6809	2.6808	2.6807
.05	2.4679	2.4679	2.4679	2.4679	2.4678	2.4678	2.4678	2.4677	2.4676	2.4676	2.4675
.06	2.2953	2.2953	2.2953	2.2953	2.2952	2.2952	2.2952	2.2952	2.2951	2.2950	2.2950
.07	2.1508	2.1508	2.1508	2.1508	2.1508	2.1508	2.1507	2.1507	2.1507	2.1506	2.1506
.08	2.0269	2.0269	2.0269	2.0269	2.0269	2.0269	2.0269	2.0268	2.0268	2.0268	2.0267
.09	1.9187	1.9187	1.9187	1.9187	1.9187	1.9187	1.9187	1.9186	1.9186	1.9186	1.9185
.1	1.8229	1.8229	1.8229	1.8229	1.8229	1.8229	1.8229	1.8228	1.8228	1.8228	1.8227
.2	1.2227	1.2226	1.2226	1.2226	1.2226	1.2226	1.2226	1.2226	1.2226	1.2226	1.2226
.3	0.9057	0.9057	0.9057	0.9057	0.9057	0.9057	0.9057	0.9057	0.9056	0.9056	0.9056
.4	7024	7024	7024	7024	7024	7024	7024	7024	7024	7024	7024
.5	5598	5598	5598	5598	5598	5598	5598	5598	5598	5598	5598
.6	4544	4544	4544	4544	4544	4544	4544	4544	4544	4544	4544
.7	3738	3738	3738	3738	3738	3738	3738	3738	3738	3738	3738
.8	3106	3106	3106	3106	3106	3106	3106	3106	3106	3106	3106
.9	2602	2602	2602	2602	2602	2602	2602	2602	2602	2602	2602
1.0	0.2194	0.2194	0.2194	0.2194	0.2194	0.2194	0.2194	0.2194	0.2194	0.2194	0.2194
2.0	489	489	489	489	489	489	489	489	489	489	489
3.0	130	130	130	130	130	130	130	130	130	130	130
4.0	38	38	38	38	38	38	38	38	38	38	38
5.0	11	11	11	11	11	11	11	11	11	11	11
6.0	4	4	4	4	4	4	4	4	4	4	4
7.0	1	1	1	1	1	1	1	1	1	1	1
8.0	0	0	0	0	0	0	0	0	0	0	0

TABLE VIII (*Continued*)

u \ β	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.10
0	9.4425	8.6319	8.0569	7.6111	7.2471	6.9394	6.6731	6.4383	6.2285	6.0388	5.8658	5.7067	5.5596	5.4228	5.2950	5.1750	5.0620	4.9553	4.8541
.000001																			
.000002																			
.000003	9.4425																		
.000004	9.4422																		
.000005	9.4413																		
.000006	9.4394																		
.000007	9.4361	8.6319																	
.000008	9.4313	8.6318																	
.000009	9.4251	8.6316																	
.00001	9.4176	8.6313	8.0569																
.00002	9.2961	8.6152	8.0558	7.6111	7.2471														
.00003	9.1499	8.5737	8.0483	7.6101	7.2470														
.00004	9.0102	8.5168	8.0320	7.6069	7.2465	6.9394	6.6731												
.00005	8.8827	8.4533	8.0080	7.6000	7.2450	6.9391	6.6730												
.00006	8.7673	8.3880	7.9786	7.5894	7.2419	6.9384	6.6729	6.4383											
.00007	8.6625	8.3233	7.9456	7.5754	7.2371	6.9370	6.6726	6.4382	6.2285										
.00008	8.5669	8.2603	7.9105	7.5589	7.2305	6.9347	6.6719	6.4381	6.2284										
.00009	8.4792	8.1996	7.8743	7.5402	7.2222	6.9316	6.6709	6.4378	6.2283										
.0001	8.3983	8.1414	7.8375	7.5199	7.2122	6.9273	6.6693	6.4372	6.2282	6.0388	5.8652	5.7067	5.5596	5.4228	5.2950				
.0002	7.8192	7.6780	7.4972	7.2898	7.0685	6.8439	6.6242	6.4143	6.2173	6.0338	5.8637	5.7059	5.5593	5.4227	5.2949	5.1750	5.0620	4.9553	
.0003	7.4534	7.3562	7.2281	7.0759	6.9068	6.7276	6.5444	6.3623	6.1848	6.0145	5.8527	5.6999	5.5562	5.4212	5.2942	5.1747	5.0619	4.9552	4.8541
.0004	7.1859	7.1119	7.0128	6.8929	6.7567	6.6088	6.4538	6.2955	6.1373	5.9818	5.8309	5.6860	5.5476	5.4160	5.2912	5.1730	5.0610	4.9547	4.8539
.0005	6.9750	6.9152	6.8346	6.7357	6.6219	6.4964	6.3626	6.2236	6.0821	5.9406	5.8011	5.6648	5.5330	5.4062	5.2848	5.1689	5.0585	4.9532	4.8530
.0006	6.8009	6.7508	6.6828	6.5988	6.5011	6.3923	6.2748	6.1512	6.0239	5.8948	5.7658	5.6383	5.5134	5.3921	5.2749	5.1621	5.0539	4.9502	4.8510
.0007	6.6527	6.6096	6.5508	6.4777	6.3923	6.2962	6.1917	6.0807	5.9652	5.8468	5.7274	5.6081	5.4902	5.3745	5.2618	5.1526	5.0471	4.9454	4.8478
.0008	6.5237	6.4858	6.4340	6.3695	6.2935	6.2076	6.1136	6.0129	5.9073	5.7982	5.6873	5.5755	5.4642	5.3542	5.2461	5.1406	5.0381	4.9388	4.8430
.0009	6.4094	6.3757	6.3294	6.2716	6.2032	6.1256	6.0401	5.9481	5.8509	5.7500	5.6465	5.5416	5.4364	5.3317	5.2282	5.1266	5.0272	4.9306	4.8368
.001	6.3069	6.2765	6.2347	6.1823	6.1202	6.0494	5.9711	5.8864	5.7965	5.7026	5.6058	5.5071	5.4075	5.3078	5.2087	5.1109	5.0133	4.9208	4.8292
.002	5.6271	5.6118	5.5907	5.5638	5.5314	5.4939	5.4516	5.4047	5.3538	5.2991	5.2411	5.1803	5.1170	5.0517	4.9848	4.9166	4.8475	4.7778	4.7079
.003	5.2267	5.2166	5.2025	5.1845	5.1627	5.1373	5.1084	5.0762	5.0408	5.0025	4.9615	4.9180	4.8722	4.8243	4.7746	4.7234	4.6707	4.6169	4.5622
.004	4.9421	4.9345	4.9240	4.9105	4.8941	4.8749	4.8530	4.8286	4.8016	4.7722	4.7406	4.7068	4.6710	4.6335	4.5942	4.5533	4.5111	4.4676	4.4230
.005	4.7212	4.7152	4.7068	4.6960	4.6829	4.6675	4.6499	4.6302	4.6084	4.5846	4.5590	4.5314	4.5022	4.4713	4.4389	4.4050	4.3699	4.3335	4.2960
.006	4.5407	4.5357	4.5287	4.5197	4.5088	4.4960	4.4814	4.4649	4.4467	4.4267	4.4051	4.3819	4.3573	4.3311	4.3036	4.2747	4.2446	4.2134	4.1812
.007	4.3882	4.3839	4.3779	4.3702	4.3609	4.3500	4.3374	4.3233	4.3077	4.2905	4.2719	4.2518	4.2305	4.2078	4.1839	4.1588	4.1326	4.1053	4.0771
.008	4.2561	4.2524	4.2471	4.2404	4.2323	4.2228	4.2118	4.1994	4.1857	4.1707	4.1544	4.1368	4.1180	4.0980	4.0769	4.0547	4.0315	4.0073	3.9822
.009	4.1396	4.1363	4.1317	4.1258	4.1186	4.1101	4.1004	4.0894	4.0772	4.0638	4.0493	4.0336	4.0169	3.9991	3.9802	3.9603	3.9395	3.9178	3.8952
.01	4.0356	4.0326	4.0285	4.0231	4.0167	4.0091	4.0003	3.9905	3.9795	3.9675	3.9544	3.9403	3.9252	3.9091	3.8920	3.8741	3.8552	3.8356	3.8150
.02	3.3536	3.3521	3.3502	3.3476	3.3444	3.3408	3.3365	3.3317	3.3264	3.3205	3.3141	3.3071	3.2997	3.2917	3.2832	3.2742	3.2647	3.2547	3.2442
.03	2.9584	2.9575	2.9562	2.9545	2.9523	2.9501	2.9474	2.9444	2.9409	2.9370	2.9329	2.9284	2.9235	2.9183	2.9127	2.9069	2.9007	2.8941	2.8873
.04	2.6807	2.6800	2.6791	2.6779	2.6765	2.6747	2.6727	2.6705	2.6680	2.6652	2.6622	2.6589	2.6553	2.6515	2.6475	2.6432	2.6386	2.6338	2.6288
.05	2.4675	2.4670	2.4662	2.4653	2.4642	2.4628	2.4613	2.4595	2.4576	2.4554	2.4531	2.4505	2.4478	2.4448	2.4416	2.4383	2.4347	2.4310	2.4271
.06	2.2950	2.2945	2.2940	2.2932	2.2923	2.2912	2.2900	2.2885	2.2870	2.2852	2.2833	2.2812	2.2790	2.2766	2.2740	2.2713	2.2684	2.2654	2.2622
.07	2.1506	2.1502	2.1497	2.1491	2.1483	2.1474	2.1464	2.1452	2.1439	2.1424	2.1408	2.1391	2.1372	2.1352	2.1331	2.1308	2.1284	2.1258	2.1232
.08	2.0267	2.0264	2.0260	2.0255	2.0248	2.0240	2.0231	2.0221	2.0210	2.0198	2.0184	2.0169	2.0153	2.0136	2.0118	2.0099	2.0078	2.0056	2.0034
.09	1.9185	1.9183	1.9179	1.9174	1.9169	1.9162	1.9154	1.9146	1.9136	1.9125	1.9114	1.9101	1.9087	1.9072	1.9056	1.9040	1.9022	1.9003	1.8983
.1	1.8227	1.8225	1.8222	1.8218	1.8213	1.8207	1.8200	1.8193	1.8184	1.8175	1.8164	1.8153	1.8141	1.8128	1.8114	1.8099	1.8084	1.8067	1.8050
.2	1.2226	1.2225	1.2224	1.2222	1.2220	1.2218	1.2215	1.2212	1.2209	1.2205	1.2201	1.2196	1.2192	1.2186	1.2181	1.2175	1.2168	1.2162	1.2155
.3	0.9056	0.9056	0.9055	0.9054	0.9053	0.9052	0.9050	0.9049	0.9047	0.9045	0.9043	0.9040	0.9038	0.9035	0.9032	0.9029	0.9025	0.9022	0.9018
.4	7024	7023	7023	7022	7022	7021	7020	7019	7018	7016	7015	7014	7012	7010	7008	7006	7004	7002	7000
.5	5598	5597	5597	5597	5596	5596	5595	5594	5594	5593	5592	5591	5590	5588	5587	5586	5584	5583	5581
.6	4544	4544	4543	4543	4543	4542	4542	4542	4541	4540	4540	4539	4538	4537	4536	4535	4534	4533	4532
.7	3738	3738	3737	3737	3737	3737	3736	3736	3735	3735	3734	3734	3733	3733	3732	3732	3731	3730	3729
.8	3106	3106	3106	3106	3105	3105	3105	3105	3104	3104	3104	3103	3103	3102	3102	3101	3101	3100	3100
.9	2602	2602	2602	2602	2601	2601	2601	2601	2601	2600	2600	2600	2599	2599	2599	2598	2598	2597	2597
1.0	0.2194	0.2194	0.2194	0.2194	0.2193	0.2193	0.2193	0.2193	0.2193	0.2193	0.2192	0.2192	0.2192	0.2191	0.2191	0.2191	0.2191	0.2190	0.2190
2.0	489	489	489	489	489	489	489	489	489	489	489	489	489	489	489	489	489	488	488
3.0	130	130	130	130	130	130	130	130	130	130	130	130	130	130	130	130	130	130	130
4.0	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38
5.0	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11
6.0	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
7.0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
8.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE VIII (*Continued*)

u \ β	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1.0
0	4.8541	4.0601	3.5054	3.0830	2.7449	2.4654	2.2291	2.0258	1.8488	1.6931	1.5550	1.4317	1.3210	1.2212	1.1307	1.0485	0.9735	0.9049	0.8420
.0001																			
.0002																			
.0003	4.8541																		
.0004	4.8539																		
.0005	4.8530																		
.0006	4.8510	4.0601																	
.0007	4.8478	4.0600																	
.0008	4.8430	4.0599																	
.0009	4.8368	4.0598																	
.001	4.8292	4.0595	3.5054																
.002	4.7079	4.0435	3.5043	3.0830	2.7449														
.003	4.5622	4.0092	3.4969	3.0821	2.7448														
.004	4.4230	3.9551	3.4806	3.0788	2.7444	2.4654	2.2291												
.005	4.2960	3.8821	3.4567	3.0719	2.7428	2.4651	2.2290												
.006	4.1812	3.8384	3.4274	3.0614	2.7398	2.4644	2.2289	2.0258											
.007	4.0771	3.7529	3.3947	3.0476	2.7350	2.4630	2.2286	2.0257											
.008	3.9822	3.6903	3.3598	3.0311	2.7284	2.4608	2.2279	2.0256	1.8488										
.009	3.8952	3.6302	3.3239	3.0126	2.7202	2.4576	2.2269	2.0253	1.8487										
.01	3.8150	3.5725	3.2875	2.9925	2.7104	2.4534	2.2253	2.0246	1.8486	1.6931	1.5550	1.4317	1.3210	1.2212	1.1307	1.0485			
.02	3.2442	3.1158	2.9521	2.7658	2.5688	2.3713	2.1809	2.0023	1.8379	1.6883	1.5530	1.4309	1.3207	1.2210	1.1306	1.0484	0.9735	0.9049	
.03	2.8873	2.8017	2.6896	2.5571	2.4110	2.2576	2.1031	1.9515	1.8062	1.6695	1.5423	1.4251	1.3177	1.2195	1.1299	1.0481	9733	9048	0.8420
.04	2.6288	2.5655	2.4816	2.3802	2.2661	2.1431	2.0155	1.8869	1.7603	1.6379	1.5213	1.4117	1.3094	1.2146	1.1270	1.0465	9724	9044	8418
.05	2.4271	2.3776	2.3110	2.2299	2.1371	2.0356	1.9283	1.8181	1.7075	1.5985	1.4927	1.3914	1.2955	1.2052	1.1210	1.0426	9700	9029	8409
.06	2.2622	2.2218	2.1673	2.1002	2.0227	1.9369	1.8452	1.7497	1.6524	1.5551	1.4593	1.3663	1.2770	1.1919	1.1116	1.0362	9657	9001	8391
.07	2.1232	2.0894	2.0435	1.9867	1.9206	1.8469	1.7673	1.6835	1.5973	1.5101	1.4232	1.3380	1.2551	1.1754	1.0993	1.0272	9593	8956	8360
.08	2.0034	1.9745	1.9351	1.8861	1.8290	1.7646	1.6947	1.6206	1.5436	1.4650	1.3860	1.3078	1.2310	1.1564	1.0847	1.0161	9510	889.	8316
.09	1.8983	1.8732	1.8389	1.7961	1.7460	1.6892	1.6272	1.5609	1.4918	1.4206	1.3486	1.2766	1.2054	1.1358	1.0682	1.0032	9411	881'	8259
.1	1.8050	1.7829	1.7527	1.7149	1.6704	1.6198	1.5644	1.5048	1.4422	1.3774	1.3115	1.2451	1.1791	1.1140	1.0505	0.9890	0.9297	0.8730	0.8190
.2	1.2155	1.2066	1.1944	1.1789	1.1602	1.1387	1.1145	1.0879	1.0592	1.0286	0.9964	0.9629	0.9284	0.8932	0.8575	8216	7857	7501	7148
.3	0.9018	0.8969	0.8902	0.8817	0.8713	0.8593	0.8457	0.8306	0.8142	0.7964	7775	7577	7369	7154	6932	6706	6476	6244	6010
.4	7000	6969	6927	6874	6809	6733	6647	6551	6446	6332	6209	6080	5943	5801	5653	5501	5345	5186	5024
.5	5581	5561	5532	5496	5453	5402	5344	5278	5206	5128	5044	4955	4860	4761	4658	4550	4440	4326	4210
.6	4532	4518	4498	4472	4441	4405	4364	4317	4266	4210	4150	4086	4018	3946	3871	3793	3712	3629	3543
.7	3729	3719	3704	3685	3663	3636	3606	3572	3534	3493	3449	3401	3351	3297	3242	3183	3123	3060	2996
.8	3100	3092	3081	3067	3050	3030	3008	2982	2953	2922	2889	2853	2815	2774	2732	2687	2641	2592	2543
.9	2597	2591	2583	2572	2559	2544	2527	2507	2485	2461	2436	2408	2378	2347	2314	2280	2244	2207	2168
1.0	0.2190	0.2186	0.2179	0.2171	0.2161	0.2149	0.2135	0.2120	0.2103	0.2085	0.2065	0.2043	0.2020	0.1995	0.1970	0.1943	0.1914	0.1885	0.1855
2.0	488	488	487	486	485	484	482	480	477	475	473	470	467	463	460	456	452	448	444
3.0	130	130	130	130	130	130	129	129	128	128	127	127	126	125	125	124	123	123	122
4.0	38	38	38	38	38	38	38	37	37	37	37	37	37	37	37	36	36	36	36
5.0	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11
6.0	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
7.0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
8.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(Continued)

u \ β	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	6.0	7.0	8.0	9.0
0	0.8420	0.4276	0.2278	0.1247	0.0695	0.0392	0.0223	0.0128	0.0074	0.0025	0.0008	0.0003	0.0001
.01													
.02													
.03	0.8420												
.04	8418												
.05	8409												
.06	8391												
.07	8360	0.4276											
.08	8316	4275											
.09	8259	4274											
.1	0.8190	0.4271	0.2278										
.2	7148	4135	2268	0.1247	0.0695								
.3	6010	3812	2211	1240	694								
.4	5024	3411	2096	1217	691	0.0392							
.5	4210	3007	1944	1174	681	390	0.0223						
.6	3543	2630	1774	1112	664	386	222	0.0128					
.7	2996	2292	1602	1040	639	379	221	127					
.8	2543	1994	1436	961	607	368	218	127	0.0074				
.9	2168	1734	1281	881	572	354	213	125	73				
1.0	0.1855	0.1509	0.1139	0.0803	0.0534	0.0338	0.0207	0.0123	0.0073	0.0025			
2.0	444	394	335	271	210	156	112	77	51	21	0.0008	0.0003	
3.0	122	112	100	86	71	57	45	34	25	12	6	2	
4.0	36	34	31	27	24	20	16	13	10	6	3	2	0.0001
5.0	11	10	10	9	8	7	6	5	4	2	1	1	0
6.0	4	3	3	3	3	2	2	2	2	1	1	0	
7.0	1	1	1	1	1	1	1	1	1	0	0		
8.0	0	0	0	0	0	0	0	0	0				

(19) $Z(\tau,\rho,\beta)$ is defined [11] by

$$Z(\tau,\rho,\beta) = K_0(\beta\rho)/K_0(\beta) + \exp(-\tau\beta^2)$$

$$\cdot \frac{2}{\pi}\int_0^\infty \frac{J_0(u\rho)Y_0(u) - Y_0(u\rho)J_0(u)}{J_0^2(u) + Y_0^2(u)} \cdot \frac{\exp(-\tau u^2)}{u^2 + \beta^2}\, u\, du$$

The function is not tabulated. It may be approximated by

a. for $\tau/\rho^2 < 0.05$,

$$Z(\tau,\rho,\beta) \approx (1/2\sqrt{\rho})\{\exp[\beta(\rho-1)]\,\text{erfc}[\beta\sqrt{\tau} + (\rho-1)/2\sqrt{\tau}] + \exp[-\beta(\rho-1)]\,\text{erfc}[-\beta\sqrt{\tau} + (\rho-1)/2\sqrt{\tau}]\}$$

b. for $\tau\beta^2 > 1$,

$$Z(\tau,\rho,\beta) \approx [W(\rho^2/4\tau, \beta\rho)]/W(1/4\tau,\beta)]$$

c. for $\tau = \infty$,

$$Z(\infty,\rho,\beta) = K_0(\beta\rho)/K_0(\beta)$$

d. for $\beta = 0$,

$$Z(\tau,\rho,0) = A(\tau,\rho)$$

(20) The gamma function, $\Gamma(x)$, is defined by

$$\Gamma(x) = \int_0^\infty \exp(-\beta)\beta^{x-1}\, d\beta, \qquad x > 0$$

and is available in tabular form [15]. A general recurrence formula for the function is

$$\Gamma(1 + x) = x\Gamma(x), \qquad \text{or} \qquad \Gamma(x) = (1/x)\Gamma(1 + x)$$

For $0 \leq x \leq 0.5$, the following relation holds

$$ln\left(\frac{\Gamma(1 - x)}{\Gamma(1 + x)}\right) \approx -0.8456x + ln\left(\frac{1 + x}{1 - x}\right)$$

III. Flow to Artesian Wells

Ground water is extracted by *pumping wells* in regions where the piezometric surface of the aquifer lies below ground surface. In regions where this surface is above ground level, ground water may be discharged by *flowing wells.* In either instance, the discharge of these wells is supplied from local storage in the main artesian aquifer, as well as from leakage, if any, originating in and/or passing through the semipervious confining beds.

The confining beds of an artesian aquifer are rarely completely impermeable. Frequently, the artesian sand is confined above and/or below by semipervious elastic clay or silt that yields significant amounts of water from storage. In certain instances, the water released from storage in these semipervious layers is much more significant in amount than that released from storage in the artesian sand they confine. These semipervious layers may in turn be overlain by water bodies in which the water levels remain more or less uniform. Such is the case when the semipervious layer reaches to the ground surface where the ground water is continuously replenished by rainfall or surface water maintaining nearly constant head on the top of the semipervious layer. A similar case occurs when the semipervious layers are over- or underlain by aquifers whose capacity for lateral flow is sufficient so that the head distribution therein is not affected by flow conditions in the main artesian aquifer in spite of the leakage from or into them. Water thus ponded at the ground surface, or accumulated in the over- and/or underlying aquifers, will be the main supply of the leakage induced by lowering the head in the main artesian aquifer.

In many cases, the semipervious layers are more or less incompressible so that water released from storage therein is very small compared to that supplied by other sources. In others, the flow through the semipervious layers is so slow that it may be neglected and the layers are assumed impermeable. The flow in each of these systems is a special case of the more general system that is described below.

A. Leaky Artesian Systems

1. Description and Assumptions

Figure 4 is a diagrammatic representation of two leaky artesian systems. Each is composed of a semipervious layer confining a main artesian aquifer resting on an impermeable bed. In one of the systems, henceforth referred to as *Case 1*, the system is overlain by a body of water in which the head distribution is not influenced by pumping in the main artesian aquifer. In the other, referred to henceforth as *Case 2*, the semipervious layer is overlain by an impermeable bed. In other systems, the main aquifer may overlie the semipervious layer which in turn may either rest on an impermeable bed or on an aquifer in which the head distribution is not influenced by flow in the main aquifer. These situations are analyzed in the same way as those of Case 2 and Case 1, respectively.

Referred to the coordinate system shown in Fig. 4, let the surface defining the bottom of the system be $f = ix$; that of the interface between the main aquifer and the semipervious layer, henceforth referred to as the "interface," be $H = b + ix$; and that of the top of the semipervious layer be $f_1 = b+b'+ix$, where b and b' are the uniform thicknesses of the main aquifer and the semipervious layer, respectively, and i is the tangent of the angle of dip of the formation.

Except where otherwise stated, the following assumptions are made: (1) the aquifers are individually elastic, homogeneous, isotropic, uniform in thickness, infinite in areal extent, and the tangent of the angle of dip i is small ($i < 0.01$); (2) the conductivity and specific storage remain constant with time and constant in the space of the layer they characterize; and (3) the wells are screened throughout the main aquifer only.

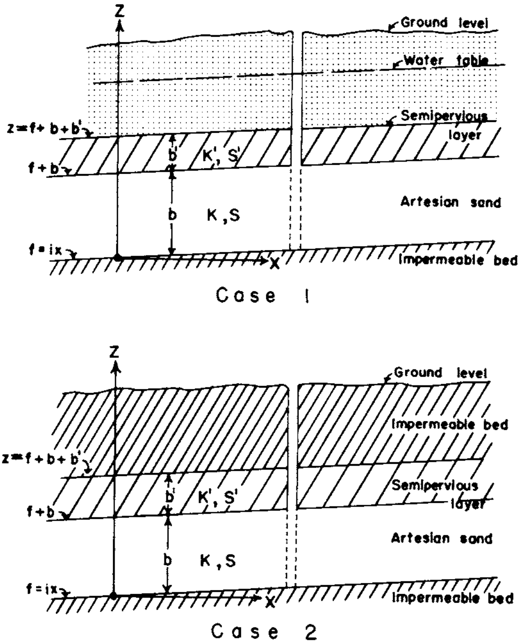

Fig. 4. Diagrammatic representation of leaky systems.

2. Vertical Motion in Semipervious Layer

For saturated flow passing from a medium of one conductivity K to another of conductivity K', a refraction in flow lines occurs such that $\tan\theta'/\tan\theta = K'/K$, where θ and θ' are, respectively, the angles which the flow lines at the interface make with the normal to the interface (Section I, D, 2, *c*). Thus, if the conductivity K' of the semipervious layer is small compared to that of the main aquifer K, as is usually the case ($K/K' > 500$), and since at the interface the vertical component of the velocity in the main aquifer is not zero, the flow in the semipervious layer will be nearly vertical; that is, the horizontal components of the flow therein are so small that in practice they may be neglected.

3. Equations of Motions in the Flow System

a. EQUATION IN SEMIPERVIOUS LAYER. Let the head distribution in the semipervious layer, if the well were not pumped, be $\varphi_{i1}(x,y,z,t)$. This head distribution is assumedly governed by Eq. (15), with $F = 0$. Let $\varphi_1(x,y,z,t)$ be the distribution of head after pumping from the main aquifer has begun. The induced *drawdown distribution* s_1 in the semipervious layer is then $s_1 = \varphi_{i1} - \varphi_1$. Having assumed nearly vertical flow in this layer—that is, the velocity components in the x- and y-directions are neglected—the substitution of φ_1 in terms of φ_{i1} and s_1 in Eq. (15), with $F = 0$, yields

$$[\partial^2\varphi_{i1}/\partial z^2 - (1/\nu')\partial\varphi_{i1}/\partial t] - [\partial^2 s_1/\partial z^2 - (1/\nu')\partial s_1/\partial t] = 0$$

in which the first bracketed term is zero, since φ_{i1} is a solution of Eq. (15), with $F = 0$ and $\partial\varphi/\partial x = \partial\varphi/\partial y = 0$; consequently, it becomes

$$\partial^2 s_1/\partial z^2 = (1/\nu')\ \partial s_1/\partial t \qquad (29)$$

in which $\nu' = K'/S_s' = K'b'/S'$, where S_s' and S' are, respectively, the specific storage and the storage coefficient of the semiconfining layer.

The leakage velocity at the interface [representing w of Eq. (16)] is given by

$$w = -K'\ \partial\varphi_1/\partial z = K'\ [\partial s_1/\partial z - \partial\varphi_{i1}/\partial z] \qquad \text{(at interface)} \qquad (30)$$

b. EQUATION IN MAIN AQUIFER. The flow in the main aquifer is a special case of that governed by Eq. (16). In the present system, $f = ix$, $H = b + ix$, $u = 0$, $\bar{F} = 0$, and $w = -K'\ \partial\varphi_1(H,t)/\partial z$. Substitution of these relations in Eq. (16) will finally give

$$\partial^2\bar{\varphi}/\partial x^2 + \partial^2\bar{\varphi}/\partial y^2 + (K'/T)\ \partial\varphi_1(H,t)/\partial z + i\ \partial[\varphi(f) - \varphi(H)]/\partial x = (1/\nu)\ \partial\bar{\varphi}/\partial t \qquad (30a)$$

in which $\nu = K/S_s = T/S$, where T and S are, respectively, the transmissivity and storage coefficient of the main aquifer. The quantity $\partial[\varphi(f) - \varphi(H)]/\partial x$ is generally small. When multiplied by i ($i < 0.01$), it becomes insignificantly small. Thus, it may for all practical purposes be neglected. In other words, the average drawdown in a vertical section of an aquifer of uniform thickness may be considered independent of the tilt of the aquifer provided that the tilt is small ($i < 0.01$).

Let $\bar{\varphi}_i(x,y,t)$ be the average head distribution in the main aquifer if the well were not pumped; assumedly $\bar{\varphi}_i$ is a solution of the preceding differential equation. Let $\bar{\varphi}(x,y,t)$ and $s(x,y,t)$ be, respectively, the average head and the induced average drawdown in the main aquifer after pumping has started. In terms of $s(s = \bar{\varphi}_i - \bar{\varphi})$ and s_1 (at interface), the preceding differential equation, after dropping the φ terms, will become

$$[\nabla^2\bar{\varphi}_i + (K'/T)\partial\varphi_{i1}(x,y,H,t)/\partial z - (1/\nu)\partial\bar{\varphi}_i/\partial t] - [\nabla^2 s + (K'/T)\partial s_1(x,y,H,t)/\partial z - (1/\nu)\partial s/\partial t] = 0$$

in which ∇^2 is the Laplacian in terms of x,y and the first braketed term is zero, since $\bar{\varphi}_i$ is a solution of Eq. (30a), with $i < 0{\cdot}01$. In polar coordinates the preceding equation then becomes

$$\partial^2 s/\partial r^2 + (1/r)\,\partial s/\partial r + (1/r^2)\,\partial^2 s/\partial \theta^2 + (K'/T)\,\partial s_1(r,H,t)/\partial z = (1/\nu)\,\partial s/\partial t \qquad (31)$$

where r and θ are the polar coordinates.

B. The Artesian-Well Flow Problem

When a single well is pumping or flowing from an effectively infinite and isotropic aquifer of uniform thickness and small tilt, the flow toward the well is essentially purely radial; that is, independent of the polar angle. The water levels at effectively infinite distances from the well maintain their initial distribution; that is, the drawdown at infinity is zero. If a *steady well*, a well of constant discharge, is pumped at the constant rate Q, the discharge, from Darcy's law is

$$Q = K\int_0^{2\pi} r_w d\theta \int_f^H [\partial\varphi(r_w,z,t)/\partial r]dz = Kr_w \int_0^{2\pi} [i(\varphi(f) - \varphi(H))\cos\theta + \partial(b\bar{\varphi})/\partial r]d\theta$$

which, if one recalls that $\int_0^{2\pi} [\partial\bar{\varphi}_i/\partial r]d\theta = 0$, will, in terms of s, finally reduce to $r_w \partial s(r_w,t)/\partial r = -Q/2\pi T$ where r_w is the effective radius of the well (Section X, E), and r is the radial distance to any point in the aquifer.

When a flowing well has been idle for a long time, the pressure head on the cap is equal to the height of a column of water reaching to the piezometric surface at the well site. As the well is opened, the piezometric surface at the site of the well drops almost instantaneously to some lower elevation. Should the so-called well loss (Section *X*, *E*) remain more or less constant, the head and consequently the drawdown at the face of the well, may be assumed to remain constant during the subsequent period of continuous flow. This drawdown, s_w, is equal to the initial pressure head at the well cap less the assumedly constant well loss. Thus, the condition at the face of a flowing well is given by $s(r_w,t) = s_w$; the initial head distribution in this case is assumed to be a function of position only.

If, as in Case 1, the semipervious layer is overlain by an aquifer maintaining a uniform pressure head, the drawdown at the top of the layer is $s_1(f_1,t) = 0$. If, on the other hand, this layer is confined by an impermeable bed, as in Case 2, the condition is $\partial s_1(f_1,t)/\partial z = 0$. In either case, the drawdown at the interface is equal to the drawdown in the main aquifer; thus, $s_1(H,t) = s(r,t)$, and the induced drawdown everywhere in the system initially is zero.

1. The Boundary-Value Problem

The artesian-well flow problem may, therefore, be described by

In semipervious layer:

$$\partial^2 s_1/\partial z^2 = (1/\nu')\partial s_1/\partial t \tag{32a}$$
$$s_1(r,z,0) = 0 \tag{32b}$$
$$s_1(r,H,t) = s(r,t) \tag{32c}$$
$$s_1(r,f_1,t) = 0, \qquad \text{for Case 1} \tag{32d}$$

or

$$\partial s_1(r,f_1,t)/\partial z = 0, \qquad \text{for Case 2} \tag{32e}$$

and

in main aquifer:

$$\partial^2 s/\partial r^2 + (1/r)\,\partial s/\partial r + (K'/T)\,\partial s_1(r,H,t)/\partial z = (1/\nu)\,\partial s/\partial t \tag{33a}$$
$$s(r,0) = 0 \tag{33b}$$
$$s(\infty,t) = 0 \tag{33c}$$
$$r_w \partial s(r_w,t)/\partial r = -Q/2\pi T, \qquad \text{for a steady well} \tag{33d}$$

or,

$$s(r_w,t) = s_w, \qquad \text{for a flowing well} \tag{33e}$$

2. Formal Solution of the Problem

Let $\bar{s}_1(r,z,p)$ and $\bar{s}(r,p)$ be the Laplace transforms of $s_1(r,z,t)$ and $s(r,t)$, respectively. Applying the transformation to Eqs. (32), by making use of Eqs. (17) to (19), will transform Eqs. (32) to

$$\partial^2 \bar{s}_1/\partial z^2 = (p/\nu')\bar{s}_1 \tag{34a}$$
$$\bar{s}_1(r,H,p) = \bar{s}(r,p) \tag{34b}$$
$$\bar{s}_1(r,f_1,p) = 0, \qquad \text{for Case 1} \tag{34c}$$

or,

$$\partial \bar{s}_1(r,f_1,p)/\partial z = 0, \qquad \text{for Case 2} \tag{34d}$$

Equation (34a) is a second-order linear differential equation whose general solution can be readily obtained. The particular solutions satisfying Eqs. (34b) and (34c) are, respectively, given for Case 1 by

$$\bar{s}_1 = \bar{s}[\sinh(b'-z)\sqrt{p/\nu'}]/\sinh b'\sqrt{p/\nu'} \tag{35}$$

with

$$\partial \bar{s}_1(r,H,p)/\partial z = -\bar{s}\sqrt{p/\nu'}\coth b'\sqrt{p/\nu'} \tag{36}$$

and for Case 2 by

$$\bar{s}_1 = \bar{s}[\cosh(b'-z)\sqrt{p/\nu'}]/\cosh b'\sqrt{p/\nu'} \tag{37}$$

with

$$\partial s_1(r,H,p)/\partial z = -\bar{s}\sqrt{p/\nu'}\tanh b'\sqrt{p/\nu'} \tag{38}$$

where sinh, cosh, tanh, and coth are the hyperbolic functions of the sine, cosine, tangent, and cotangent, respectively.

If the Laplace transformation is applied to Eqs. (33) through use of Eqs. (17) to (19), observing that the transform of a constant A is A/p, Eqs. (33) will, after using Eqs. (36) and (38), become

for steady wells:

Case 1.

$$\partial^2\bar{s}/\partial r^2 + (1/r)\partial\bar{s}/\partial r - \lambda^2\bar{s} = 0 \tag{39a}$$

$$\bar{s}(\infty,p) = 0 \tag{39b}$$

$$r_w\partial\bar{s}(r_w,p)/\partial r = -\ Q/2\pi Tp \tag{39c}$$

Case 2. Equations (39), with λ_1 replacing λ;

for flowing wells:

Case 1.

$$\text{Eq. (39a)} \tag{40a}$$

$$\bar{s}(\infty,p) = 0 \tag{40b}$$

$$\bar{s}(r_w,p) = s_w/p \tag{40c}$$

Case 2. Equations (40), with λ_1 replacing λ, where

$$\begin{aligned} \lambda^2 &= p/\nu + (K'/b'T)(b'\sqrt{p/\nu'}) \coth b'\sqrt{p/\nu'} \\ \lambda_1{}^2 &= p/\nu + (K'/b'T)(b'\sqrt{p/\nu'}) \tanh b'\sqrt{p/\nu'} \end{aligned} \tag{41}$$

Equation (39a) is the modified Bessel equation of zero order whose general solution (Section II, B, Example 9) is

$$\bar{s} = c_1K_0(\lambda r) + c_2I_0(\lambda r)$$

Since $K_0(\infty) = 0$, and $I_0(\infty) = \infty$, then for all cases, the value of c_2 is found (by using the common condition $\bar{s}(\infty,p) = 0$) equal to zero. Since $\partial K_0(\lambda r)/\partial r = -\lambda K_1(\lambda r)$, the value of c_1 for the cases of a steady well is obtained by using Eq. (39c) as $Q/2\pi Tp\lambda r_wK_1(\lambda r_w)$. For the cases of a flowing well, c_1 is obtained by using Eq. (40c) as $s_w/pK_0(\lambda r_w)$. Thus, after substituting these values of c_1 and c_2 and applying the inverse Laplace transformation, the formal solutions for the different cases of the problem are

for steady wells:

Case 1.

$$s(r,t) = (Q/2\pi T)L^{-1}\{K_0(\lambda r)/p\lambda r_wK_1(\lambda r_w)\} \tag{42}$$

Case 2.

$$\text{Eq. (42), with } \lambda_1 \text{ replacing } \lambda \tag{43}$$

for flowing wells:

Case 1.

$$s(r,t) = s_w L^{-1}\{K_0(\lambda r)/pK_0(\lambda r_w)\} \tag{44}$$

Case 2.

$$Eq.\ (44),\ \text{with } \lambda_1 \text{ replacing } \lambda \tag{45}$$

The inverse Laplace transforms in Eqs. (42) to (45) are not available. Consequently, exact solutions to these cases are yet to be found. Asymptotic solutions can be obtained, however.

3. *Asymptotic Solutions*

The behavior of $s(t)$ at large and small values of time may be determined, respectively, from the behavior of $\bar{s}(p)$ for small and large values of the transform parameter p. Thus, the solutions for long-time periods and those for short-time periods correspond to those obtained from Eqs. (42) to (45) as p becomes small and large, respectively.

a. SOLUTIONS FOR LONG TIMES. As p becomes small, the arguments of the hyperbolic functions of Eqs. (41) become relatively small. Since for $x < 0.4$, $x \coth x \approx 1 + x^2/3$ and $\tanh x \approx x$, it follows that for $b'\sqrt{p/\nu'} < 0.4$ (of the order $t > 2b'S'/K'$), λ and λ_1 of Eq. (41) become, respectively,

$$\begin{aligned} \gamma^2 &= \delta_1 p/\nu + 1/B^2 \\ \gamma_1{}^2 &= \delta_2 p/\nu \end{aligned} \tag{46}$$

where

$$\delta_1 = 1 + S'/3S,\ \ \delta_2 = 1 + S'/S,\ \text{ and }\ B^2 = T/(K'/b').$$

Also, since for $x < 0.1$, $K_1(x) \approx 1/x$, then for $\gamma r_w < 0.1$ (of the order $t > 30\,\delta_1 r_w^2/\nu[1 - (10r_w/B)^2]$, with $r_w/B < 0{\cdot}1$) and $\gamma_1 r_w < 0.1$ (of the order $t > 30\,\delta_2 r_w^2/\nu$), the following approximations may be made:

$$\gamma r_w K_1(\gamma r_w) \approx 1 \qquad \text{and} \qquad \gamma_1 r_w K_1(\gamma_1 r_w) \approx 1 \tag{47}$$

By making these approximations and using the short tables of transforms (Section II, 1, *b*) Eqs. (42) to (45) become, respectively:

for steady wells:

Case 1, for $t >$ *both* $2b'S'/K'$ *and* $30\,\delta_1 r_w^2/\nu[1-(10r_w/B)^2]$, with $r_w/B < 0{\cdot}1$,

$$s(r,t) = (Q/4\pi T)\,W(\delta_1 u,\ r/B) \tag{48}$$

Case 2, for $t >$ *both* $2b'S'/K'$ *and* $30\,\delta_2 r_w^2/\nu$,

$$s(r,t) = (Q/4\pi T)\,W(\delta_2 u) \tag{49}$$

for flowing wells, $t > 2b'S'/K'$:

Case 1.

$$s(r,t) = s_w Z(\tau/\delta_1,\rho,r_w/B) \tag{50}$$

Case 2.

$$s(r,t) = s_w A(\tau/\delta_2,\rho) \tag{51}$$

where

$$u = r^2/4\nu t, \qquad \tau = \nu t/r_w^2, \qquad \text{and} \qquad \rho = r/r_w$$

b. SOLUTIONS FOR SHORT TIMES. As p becomes large, the values of the functions $\coth b'\sqrt{p/\nu'}$ and $\tanh b'\sqrt{p/\nu'}$ approach unity. Since for $x > 3$, $\coth x \approx \tanh x \approx 1$, it follows that for $b'\sqrt{p/\nu'} > 3$ (of the order $t > b'S'/10K'$), λ and λ_1 of Eqs. (41) both approach the value

$$\gamma_0^2 = p/\nu + (K'/T)\sqrt{p/\nu'} \tag{52}$$

Consequently, the solution to both Cases 1 and 2 will be the same.

Theoretically, the required solutions will be given by Eqs. (42) to (45), with γ_0 replacing λ and λ_1. But the inverse transforms of the resulting expressions are not available. A solution, however, may be obtained for the cases of a steady well, provided the radius of the well can be assumed as vanishingly small.

Because the well radius is small compared to other dimensions of practical interest, this assumption does not seriously affect the drawdown distribution around the well except at points very close to the well during the early period of pumping.

For wells of small radii ($r_w \to 0$) the limit of $\gamma_0 r_w K_1(\gamma_0 r_w) = 1$. Consequently Eqs. (42) and (43), with γ_0 of Eq. (52) replacing both λ and λ_1, will give the solution for

steady wells in Cases 1 and 2 as

$$s = (Q/4\pi T)H(u,\beta) \tag{53}$$

in which

$$u = r^2/4\nu t \qquad \text{and} \qquad \beta = (r/4B)\sqrt{S'/S} = r\sqrt{K'S_s'/16TS}$$

C. Drawdown and Yield Formulas for Steady or Constant-Discharge Wells

Expressions for the drawdown, the rate and volume of yield from storage in the main aquifer, and the rate and volume of induced leakage are given below. The reader is referred to Section II, C for the definition, approximation, and tables of the functions appearing in the following and subsequent solutions. Evaluation of the functions outside the range of their tabular value, may be effected, in general, by using their approximate forms.

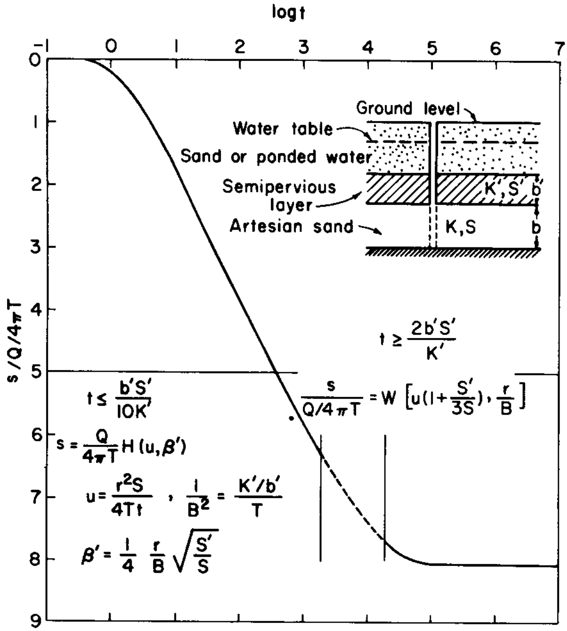

FIG. 5. Time-drawdown variation in the main aquifer owing to a steady well in the flow system shown.

1. Wells in Leaky Systems with Storage in Semipervious Layer

a. CASE 1, FIG. 4. The flow formulas for this case are as follows:

(*i*) *Long-time solutions, $t >$ both $2b'S'/K'$ and* $30\,\delta_1 r_w^2/\nu[1 - (10r_w/B)^2]$ and $r_w/B < 0{\cdot}1$. In this range of time the solutions are obtained as follows:

Unsteady drawdown equation: From Eq. (48), the equation of drawdown is

$$s = (Q/4\pi T)W(\delta_1 u, r/B) \tag{54}$$

another form of which, useful for large values of time, may be written [28] as

$$s = (Q/4\pi T)\{2K_0(r/B) - W(q/\delta_1, r/B)\}$$

in which $u = r^2/4\nu t$, $q = \nu t/B^2$, and $(1/B^2) = (K'/b')/T$, where (K'/b') is a constant characteristic of the semipervious layer which is a measure of the ability of this layer to transmit vertical leakage. It is known as the *coefficient of leakage* (or *leakance*) and is defined as the rate of flow that crosses a unit area of the interface between the main aquifer and the semipervious layer, if the difference between the heads at the top and bottom of the semipervious layer is unity.

These drawdown equations are valid for any well radius r_w (not necessarily small, as is usually assumed), provided that $t > 30\,\delta_1 r_w^2/\nu[1 - (10r_w/B)^2]$, with $r_w/B < 0{\cdot}1$.

Steady drawdown equations: As time becomes effectively large, the yield of the well will be sustained almost entirely by leakage passing through the semi-pervious layer; thus, a steady state of flow will be essentially realized. As $t \to \infty, p \to 0$, and λ of Eq. (42) becomes $1/B$. Consequently, the steady drawdown, from Eq. (42), with $\lambda = 1/B$, is

$$s = (Q/2\pi T)[K_0(r/B)/(r_w/B)K_1(r_w/B)]$$

which, since in practice $r_w/B < 0.01$ and since $xK_1(x) \approx 1$ for $x < 0.1$, may for practical computations be written as

$$s = (Q/2\pi T)K_0(r/B) \tag{55}$$

Figure 5 shows the variation of drawdown with the logarithm of time for the flow system of Case 1.

Rate and volume of yield from storage in main aquifer: From the definition of storage coefficient, $\partial q_s/\partial \rho = 2\pi r S\, \partial s/\partial t$, where q_s is the rate of yield from storage in the main aquifer. From Eq. (54), $\partial s/\partial t = (Q/4\pi T)(1/t)\exp(-r^2\,\delta_1/4\nu t - \nu t/\delta_1 B^2)$. Consequently,

$$q_s = (QS/4\pi Tt)\exp(-\nu t/\delta_1 B^2)\int_0^\infty \exp(-r^2\delta_1/4\nu t)(2\pi r\, dr)$$

or

$$q_s = (Q/\delta_1)\exp(-\nu t/\delta_1 B^2) \tag{56}$$

The total volume of yield (V_s) from storage in the main aquifer within any period when $t >$ both $2b'S'/K'$ and $30\,\delta_1 r_w^2/\nu[1 - (10r_w/B)^2]$, is obtained by integrating Eq. (56) with respect to t between the limits of the period. Thus,

$$V_s = Q(B^2/\nu)[\exp(-\nu t_1/\delta_1 B^2) - \exp(-\nu t_2/\delta_1 B^2)] \tag{57}$$

where $t_2 > t_1 >$ both $2b'S'/K'$ and $30\,\delta_1 r_w^2/\nu[1 - (10r_w/B)^2]$; t_1 and t_2 are the limits of the period in question.

Rate and volume of induced leakage: The rate q_L and total volume V_L of induced leakage are, respectively, given by

$$q_L = Q - q_s \quad \text{and} \quad V_L = Q(t_2 - t_1) - V_s \tag{58}$$

where q_s and V_s are as given by Eqs. (56) and (57), respectively.

(*ii*) *Short-time solutions, $t < S'b'/10K'$.* Within this range of time the solutions are as follows.

Drawdown equation: From Eq. (53), the equation of drawdown is

$$s = (Q/4\pi T)H(u,\beta) \tag{59}$$

where $\beta = (r/4\text{B})\sqrt{S'/S}$.

Figure 6 shows the variation of drawdown with the logarithm of time at different radial distances from the center of the well during relatively short periods of pumping for the system shown in the figure.

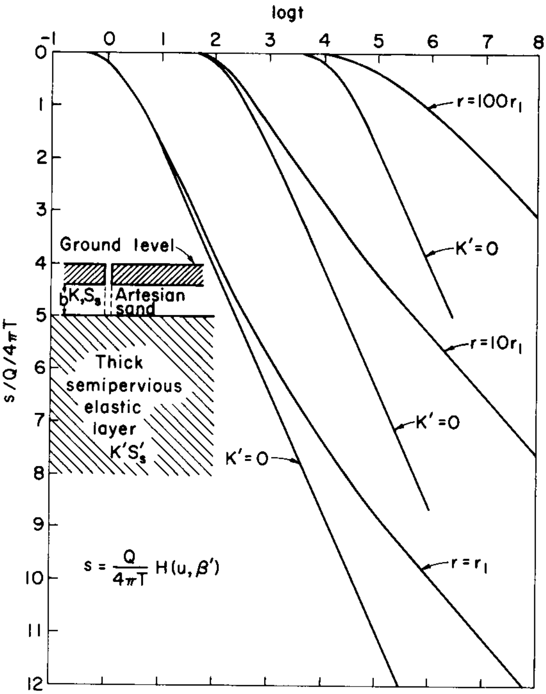

FIG. 6. Variation of drawdown with time and distance in the main aquifer due to a steady well in the flow system shown.

The rate and total volume of yield from storage in the main aquifer and the rate and total volume of induced leakage, obtained in a manner similar to that used for long times, are, respectively, given by

$$\begin{aligned} q_s &= Q \exp(nt) \operatorname{erfc}(\sqrt{nt}) \\ V_s &= Qt\{2/\sqrt{n\pi t} - (1/nt)[1 - \exp(nt)\operatorname{erfc}(\sqrt{nt})]\} \\ q_L &= Q - q_s, \qquad V_L = Qt - V_s \end{aligned} \tag{60}$$

where $n = (K'/b')(S'/S^2) = K'S_s'/S^2$ and t is time since pumping started, provided $t < S'b'/10K'$.

(*iii*) *Solutions for intermediate time.* Expressions for the intermediate range of time are given neither for Case 1 nor for Case 2 (presented subsequently). The two asymptotic solutions of each can, however, be used to obtain an approximate solution for the intermediate range by graphical interpolation on semilogarithmic paper. For example, values of q_s/Q, obtained from the two asymptotic expressions, are plotted against values of nt (or t) on semilogarithmic paper, with nt (or t) plotted on the logarithmic scale. A smooth curve joining the two branches of the calculated curve is then constructed by inspection, from which approximate values of q_s/Q can be obtained. Similarly, approximate values for s and V_s in the intermediate range of time may be obtained.

b. CASE 2, FIG. 4. The flow formulas for this case are special cases of those of Case 1. Comparing the drawdown equations of Cases 1 and 2 as given by Eqs. (48) and (49), it is clear that if the parameters δ_1 and B in the solution of Case 1 are replaced, respectively, by δ_2 and ∞, the resulting equation will be the solution to Case 2. Consequently, the flow formulas for Case 2 are the limit of their counterparts of Case 1 as $B \to \infty$ and $\delta_1 \to \delta_2$. Thus, the solutions are as follows.

(*i*) *Long-time solutions,* $t >$ *both* $2b'S'/K$ *and* $30\ \delta_2 r_w^2/\nu$. The drawdown, the rate and volume of yield from storage in the main aquifer, and the rate and volume of induced leakage are, respectively, given by

$$\begin{aligned} s &= (Q/4\pi T) W(\delta_2 u) \\ q_s &= Q/\delta_2, \qquad V_s = Q(t_2 - t_1)/\delta_2 \\ q_L &= Q(1 - 1/\delta_2), \qquad V_L = Q(t_2 - t_1)(1 - 1/\delta_2) \end{aligned} \tag{61}$$

Figure 7a shows the variation of drawdown with the logarithm of time for different values of specific storage. In Fig. 7b the variation is shown at a more distant well than that of Fig. 7a.

(*ii*) *Short-time solutions,* $t < b'S'/10K'$. During this time range the flow formulas are the same as those of Case 1 and hence given by Eqs. (59) and (60).

c. LEAKY SYSTEM OF AN INDEFINITELY THICK SEMIPERVIOUS LAYER. The flow formulas for this case are special cases of their counterparts of Cases 1 or 2 for $t < b'S'/10K'$. As $b' \to \infty$, the time criterion will be $t < \infty$, or the whole range of time. Consequently, the required equations are those given by Eqs. (59) and (60), valid for the whole range of time.

2. *Wells in Leaky Systems without Storage in Semipervious Layer.*

If the semipervious layer of a leaky artesian system is more or less incompressible so that the leakage derived from storage in the semipervious layer is negligible in comparison with that derived from other sources, the specific storage and hence the storage coefficient of this layer may, for all practical purposes, be assumed zero. Consequently, the limit of the flow formulas of Case 1 as $S' \to 0$ will give their counterparts for the present leaky system. As $S' \to 0$, $\delta_1 \to 1$, and the time criterion becomes $t > 30\ r_w^2/\nu[1 - (10r_w/B)^2]$, with $r_w/B < 0{\cdot}1$.

a. DRAWDOWN EQUATIONS. For $t > 30r_w^2/\nu[1 - (10r_w/B)^2]$ and $r_w/B < 0{\cdot}1$,

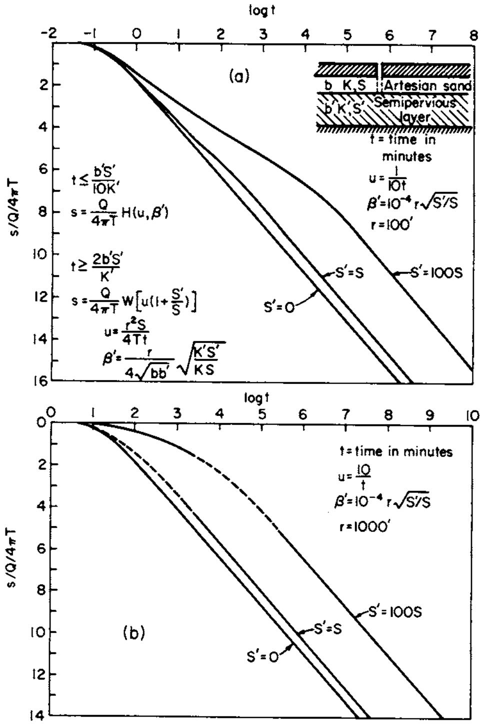

FIG. 7. Variation of drawdown in the main aquifer with time, distance, and specific storage due to a steady well in the flow system shown.

the *unsteady drawdown* equation, from Eq. (54), with $\delta_1 = 1$, is readily obtained as

$$s = (Q/4\pi T)W(u, r/B) \tag{62}$$

This equation, generally known as the *Hantush-Jacob formula*, is valid for all values of r_w (not necessarily small, as is usually assumed), provided $t > 30r_w^2/\nu[1 - (10r_w/B)^2]$ and $r_w/B < 0{\cdot}1$.

Depending on the value of r_w/B, the *steady drawdown distribution* is given by Eq. (55) or by that preceding Eq. (55).

Figure 8 compares the time-drawdown variation in leaky systems.

b. YIELD EQUATIONS. For $t > 30r_w^2/\nu[1 - (10r_w/B)^2]$ and $r_w/B < 0{\cdot}1$, the rate and volume of yield from storage in main aquifer and the rate and volume of induced leakage are given by Eqs. (56) to (58) with $\delta_1 = 1$.

3. *Wells in Nonleaky Aquifers*

Should the flow in the semipervious layers be so slow as to be negligible in comparison with that in the main aquifer, a nonleaky artesian aquifer results.

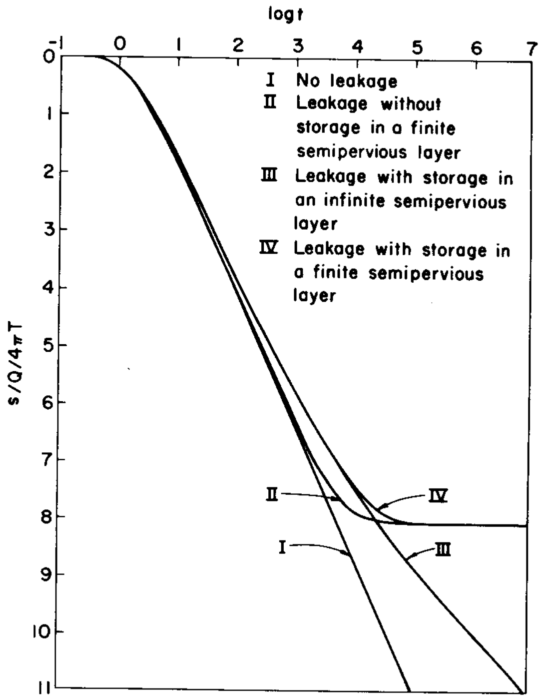

FIG. 8. Time-drawdown variation in the main aquifer of a leaky system with and without storage in the semipervious layer.

In such instances, these layers may be assumed impermeable ($K'=0$). The yield of the well is virtually sustained by water derived from storage in the main aquifer. Thus the solution to this Case may be obtained from Eq. (62) as $K' \to 0$. As $K' \to 0$, $B \to \infty$, whence $W(u, r/B) \to W(u,0) = W(u)$ (Section II, C, 17). Consequently, the drawdown distribution, form Eq. (62), will be

$$s = (Q/4\pi T)W(u) \tag{63}$$

which is the well-known *Theis formula*; it is valid for any value of r_w, provided $t > 30r_w^2/\nu$.

In most cases of artesian flow, this time criterion applies almost from the start of pumping. For example, if $S = 0.001$ (a high storage coefficient for artesian conditions), $T = 0.001$ ft²/sec (a low transmissivity), and $r_w = 10$ ft (a large well radius), Eq. (63) as well as others governed by this time criterion will be applicable for $t > (30)(100)/(0.001/0.001)$, or $t > 3000$ sec, or 50 min; the equation, of course, fails to apply within the first 50 min of the period of pumping.

An exact solution applicable for all values of t and r_w is available [10, p.338]. It is obtained as follows: For $K' = 0$, λ of Eq. (41) is equal to $\sqrt{p/\nu}$. Consequently, Eq. (42) becomes

$$s(r,t) = (Q/2\pi T)L^{-1}\{K_0(r\sqrt{p/\nu})/p(r_w\sqrt{p/\nu})K_1(r_w\sqrt{p/\nu})\}$$

or, from the short table of transforms (Section, II, 1, b), it is

$$s = (Q/4\pi T)S(\tau,\rho) \tag{64}$$

in which

$$\tau = \nu t/r_w^2 \qquad \text{and} \qquad \rho = r/r_w$$

The function $S(\tau,\rho)$ is not sufficiently tabulated for general practical use (Table VI); it is, however, sufficiently tabulated for the important special case of the *drawdown at the face of the well*; that is for $\rho = 1$.

D. Recovery Equations for Steady Wells

If a well is pumped at a constant rate for a known period of time and then shut down, the residual drawdown s' (the drawdown during recovery) is (Section II, Example 10) given by

$$s' = s(t) - s(t') \tag{65}$$

in which t and t' are the times reckoned, respectively, from the commencement and end of pumping; that is, $t = t_0 + t'$ and t_0 is the period of pumping. Thus, the recovery equation corresponding to any of the drawdown equations, given in Section III, C, can be formulated readily subject to the same time criteria. For example, the recovery equation for a well in a *leaky system* without storage in semipervious layer will, from Eq. (62), be given by

$$s' = (Q/4\pi T)[W(u, r/B) - W(u', r/B)] \tag{66}$$

and for a well in a *nonleaky aquifer*, from Eq. (64), is given by

$$s' = (Q/4\pi T)[S(\tau,\rho) - S(\tau',\rho)]$$

or if $t' > 30r_w^2/\nu$, will, from Eq. (63), be given by

$$s' = (Q/4\pi T)[W(u) - W(u')] \tag{67}$$

in which u' and τ' are the values of u and τ after replacing t by t'.

For $u' < 0.05$ or $t' > 5r^2/\nu$, Eq. (67) may be approximated (see approximation of $W(u)$ and $W(u')$ for this range of u and u') by the Theis recovery equation, namely,

$$s' = (Q/4\pi T)\ln(t/t') \tag{68}$$

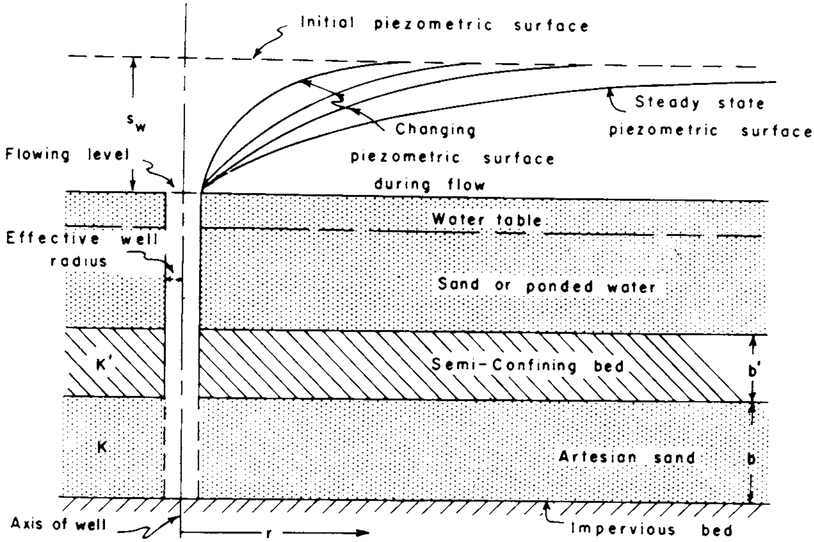

FIG. 9. Diagrammatic representation of a flowing well in a leaky aquifer.

E. DRAWDOWN AND YIELD FORMULAS FOR FLOWING WELLS

1. *Wells in Leaky Systems with Storage in Semipervious Layer*

Only long-time solutions are available for this flow system. These are

a. CASE 1, FIG. 4, $t > 2b'S'/K'$. In this range of time the flow formulas are obtained as follows:

Drawdown equations: From Eq. (50), the equation of unsteady drawdown is

$$s = s_w Z(\tau/\delta_1, \rho, r_w/B) \tag{69}$$

from which as $t \to \infty$, the drawdown during the steady state (Section II, C, 19, *d*), is

$$s = s_w K_0(r/B)/K_0(r_w/B)$$

Figure 9 diagrammatically shows flow conditions of a flowing well in a leaky aquifer.

Yield equations: The decline of the discharge of the well with time is given by $Q = -2\pi r_w T \, \partial s(r_w,t)/\partial r$. If $\partial s/\partial r$ is obtained from Eq. (69), then evaluated at $r = r_w$, the result after mathematical reduction [11] will be

$$Q(t) = 2\pi T s_w G(\tau/\delta_1, r_w/B) \tag{70}$$

The rate of yield derived from storage in the main aquifer is given (Section III, C, 1, *a*) by

$$q_s = 2\pi S \int_{r_w}^{\infty} (r \, \partial s/\partial t) \, dr$$

from which, by using Eq. (69) and after mathematical reduction, the result will be

$$q_s = (2\pi T s_w/\delta_1)\exp[-\tau(r_w/B)^2/\delta_1]G(\tau/\delta_1)$$

Consequently, the rate of induced leakage is

$$\begin{aligned} q_L &= Q(t) - q_s \\ &= 2\pi T s_w\{G(\tau/\delta_1, r_w/B) - (1/\delta_1)\exp[-(\tau/\delta_1)(r_w/B)^2]G(\tau/\delta_1)\} \end{aligned}$$

The total yield of the well V within a given period of time, provided its lower limit t_1 is greater than $2b'S'/K'$, is obtained by integrating Eq. (70) with respect to time between t_1 and t_2, where t_2 is the end of the period in question. Thus,

$$V = 2\pi\delta_1 S r_w^2 s_w \int_{\tau_1/\delta_1}^{\tau_2/\delta_1} G(\tau, r_w/B)\, d\tau$$

or

$$V = 2\pi\delta_1 S r_w^2 s_w\{G_t(\tau_2/\delta, r_w/B) - G_t(\tau_1/\delta_1, r_w/B)\}$$

where

$$G_t(y, \beta) = \int_0^y G(\tau, \beta) d\tau$$

is a function that can be easily tabulated for a practical range of y and β by using Table II for the function $G(\tau, \beta)$. Obviously, $G_t(0, \beta) = 0$.

That part of the total yield of the well that is from storage in the main aquifer V_s is similarly obtained as

$$V_s = 2\pi S r_w^2 s_w\{G_s(\tau_2/\delta_1, r_w/B) - G_s(\tau_1/\delta_1, r_w/B)\}$$

where

$$G_s(y, \beta) = \int_0^y \exp(-\tau\beta^2)G(\tau)\, d\tau$$

which can also be easily tabulated for a practical range of y and β by using tables for the exponential function and the function $G(\tau)$. Obviously also,

$$G_s(0, \beta) = 0 \qquad \text{and} \qquad G_s(\tau, 0) = G_t(\tau, 0)$$

The total yield of the well, that is from induced leakage, is $V_L = V - V_s$

b. CASE 2, FIG. 4, $t > 2b'S'/K'$. As previously noted (Section III, C, 1, *b*), the flow formulas for this case are the limit ($B \to \infty$ and $\delta_1 \to \delta_2$) of their counterparts in Case 1. Consequently, the required equations are

$$\begin{aligned} s &= s_w A(\tau/\delta_2, \rho) \\ Q(t) &= 2\pi T s_w G(\tau/\delta_2) \\ q_s &= Q(t)/\delta_2, \qquad q_L = Q(t)(1 - 1/\delta_2) \\ V &= 2\pi\delta_2 S r_w^2 s_w\{G_t(\tau_2/\delta_2, 0) - G_t(\tau_1/\delta_2, 0)\} \\ V_s &= V/\delta_2, \qquad V_L = V(1 - 1/\delta_2) \end{aligned} \tag{71}$$

Observe that $G_t(\tau,0) = G_s(\tau,0)$.

2. Wells in Leaky Systems without Storage in Semipervious Layer

As noted in Section III, C, 2, the flow formulas for this flow system are the limit of their counterparts in Case 1, as $S' \to 0$; that is, for $\delta_1 = 1$. Thus, the required solutions are given by the equations of Section III, E, 1, after replacing δ_1 by unity. These expressions are valid for $t > 2b'S'/K'$, or $t > 0$, since $S' = 0$.

3. *Wells in Nonleaky Aquifers*

If S' is made equal to zero in the flow system of Case 2, a nonleaky system results. The flow equations for a nonleaky aquifer are, therefore, given by Eqs. (71), with $\delta_2 = 1$. The expressions are valid for the whole range of time; that is, for $t > 0$.

F. Drawdown Formulas for Wells of Variable Discharge

Frequently, the discharge of a pumping well declines with time during the early period of pumping. Eventually, the discharge may attain a constant rate. Such a discharge variation may be due to the self-adjustment of a constant-speed pump to the declining head in the well in accordance with the pump head-discharge characteristics and the hydraulic properties of the aquifer. The period of decreasing discharge may range from a few tens of minutes to several days. Frequently, the observed discharge-time variation may be fitted closely to a simple mathematical curve, using curve-fitting techniques.

1. *Wells in Nonleaky and Leaky Systems without Storage in Semipervious Layer*

(*a*) exponentially decreasing discharge. The observed discharge variation may be represented by

$$Q_t = Q_s[1 + \delta \exp(-t/t^*)]$$

in which Q_t is the variable discharge of the well, Q_s is the eventual constant discharge, and δ and t^* are constants that depend on the aquifer and pump characteristics; Q_s, δ, and t^* are obtained empirically.

For this type of discharge variation, the drawdown equation for a well in a *leaky system without storage in semipervious layer* may be shown, using a procedure similar to that followed in Example 10, to be expressed as follows:

For $\nu t^*/B^2 > 1$, the drawdown equation is given by

$$s = (Q_s/4\pi T)\{W(u,r/B) + \delta e^{-t/t^*} W(u,\sqrt{(r/B)^2 - 4u^*})\}$$

and for $\nu t^*/B^2 < 1$, the equation is

$$s = (Q_s/4\pi T)\{W(u,r/B) + \delta I(u,u^*,r/B)\} \qquad (71a)$$

in which

$$u = r^2/4\nu t, \qquad u^* = r^2/4\nu t^*, \qquad B = \sqrt{T/(K'/b')},$$

and

$$I(u,u^*,r/B) = \exp(-u^*/u) \int_u^\infty \exp\left(-y + \frac{4u^* - (r/B)^2}{4y}\right) \frac{dy}{y}$$

The function I is not available in tabular form. Use may, however, be made of a tabulation [61] of a related function to construct a partial tabulation for I. In the range where $u \leq 0{\cdot}01$, the function may be approximated by

$$I(u,u^*,r/B) \simeq \exp\left(-\frac{u^*}{u}\right)\{W(0.1,r/B) + (1-\beta)\,[E_i(\beta/u) - E_i(10\beta)] + u\exp(\beta/u) - 0.1\exp(10\beta)\}$$

in which $\beta = u^* - (r/2B)^2$ and $E_i(x) = \int_{-\infty}^{x} (e^y/y)dy$ is the exponential integral, tabular values of which are available [15].

The equation of drawdown for such wells in *nonleaky aquifers* is given by Eq. (71a), with $1/B = 0$.

b. HYPERBOLICALLY DECREASING DISCHARGE. If the observed discharge can be represented by

$$Q_t = Q_s[1 + \delta/(1+t/t^*)]$$

the drawdown equation for a well in a *leaky system without storage in the semipervious layer* may be shown to be given by

$$s = (Q_s/4\pi T)\{W(u,r/B) + \delta I^*(u,t/t^*,r/B)\}$$

and for a well in a *nonleaky aquifer* by

$$s = (Q_s/4\pi T)\left\{W(u) + \frac{\delta}{1+(t/t^*)}\exp\left(-\frac{(t/t^*)u}{1+(t/t^*)}\right)W\left(\frac{u}{1+(t/t^*)}\right)\right\},$$

where

$$I^*(u,t/t^*,r/B) = \frac{1}{1+t/t^*}\int_u^{\infty}\exp\left(-y - \frac{(r/B)^2}{4y}\right)\frac{dy}{[y-(t/t^*)u/(1+t/t^*)]}$$

and other symbols have been defined. This function is not available in tabular form. Tabulation of the function for a practical range of the parameters involved is not difficult, however.

c. If the discharge decreases in accordance with

$$Q_t = Q_s\left[1 + \frac{\delta}{\sqrt{1+t/t^*}}\right]$$

the drawdown equation for a well in a *non-leaky aquifer* can be obtained as

$$s = (Q_s/4\pi T)\left\{W(u) + \frac{\delta}{\sqrt{1+t/t^*}}\exp\left(-\frac{u}{2(1+t^*/t)}\right)\left[K_0\left(\frac{u}{2(1+t^*/t)}\right) + E\left(\frac{u}{2(1+t^*/t)}, \frac{u}{\sqrt{1+t/t^*}}\right) - \cosh^{-1}(1+2t^*/t)\right]\right\}$$

in which

$$E(a,x) = \int_o^x [1-\exp(-\sqrt{\beta^2+a^2})\frac{d\beta}{\sqrt{\beta^2+a^2}}$$

and $\cosh^{-1}$ is the inverse hyperbolic cosine, tabular values for which are available [15]. The function $E(a,x)$ is tabulated for a wide range of a and x [63]. For $u \leq 0.1$, or $\sqrt{a^2 + x^2} < 0.1$, the function may be approximated by

$$E(a,x) \approx x \approx u/\sqrt{1 + t/t^*}$$

G. General Remarks and Examples

The effect of storage in the semipervious layer of a leaky artesian system on the drawdown induced by a steady well is depicted in Figs. 5 to 8. If the ratio of the storage coefficient of the semipervious layer to that of the main aquifer is small ($S'/S < 0.01$), the effect of storage in the semipervious layer on the drawdown is very small in flow systems such as those in Figs. 5 and 7a, and may be neglected. In such instances, the limiting drawdown formulas—namely, the Hantush-Jacob and the Theis formulas—will describe fairly accurately the flow in the respective flow systems. For relatively larger values of S'/S, the drawdown deviates considerably from that given by these limiting formulas.

Semilogarithmic time-drawdown curves of a flow system such as that shown in Fig. 5 have the same general trend as the curves for a system having negligible storage in the semipervious layer (Hantush-Jacob formula). Also, for relatively large distances in a flow system of high storage in the semipervious layer (Figs. 6 and 7b), such time-drawdown curves have the same general appearance as curves for systems having no leakage (Theis formula). Thus, observational drawdown trends may be mistakenly assumed to be represented by the limiting formulas, in which event analysis of such data for determining formation coefficients (S, T, and K'/b'), based upon the limiting formulas, will give erroneous and, in many instances, unreasonable results. This, of course, leads to the vexations which arise when attempts are made to force the application of formulas to cases to which they do not apply.

Example 11. The estimated formation coefficients of a leaky system such as that of Case 1 are $T = 0.05$ ft²/sec, $K'/b' = 5 \times 10^{-8}\text{sec}^{-1}$, $S = 10^{-4}$, and $S' = 16 \times 10^{-4}$. A well 24 in. in diameter is to be pumped continuously at a constant rate for three days every two weeks. What is the expected yield of the well if the drawdown in the well (neglecting well losses) is not to exceed 50 ft at the end of the pumping period? What is the corresponding yield if the artesian system is regarded as nonleaky? The well is not influenced by nearby pumping wells.

The discharge Q is to be computed for $s = 50$ at $t = t_0 = 3$ days $= 25.92 \times 10^4$ sec. Thus, $t_0 > (2)S'/(K'/b') = 6.4 \times 10^4$. For $\delta_1 = 1 + S'/3S = 6.33$ and $\nu = T/S = 500$ ft²/sec, $r_w = 1$, and $B = \sqrt{T/(K'/b')} = 1000$ ft, t_0 is greater than $30\delta_1 r_w^2/\nu[1 - (10r_w/B)^2]$. Consequently, Eq. (54) applies, or $Q = 4\pi Ts/W(u\delta_1, r_w/B)$. For $\delta_1 u = \delta_1 r_w^2/4\nu t_0 = (6.33)\,(1.93 \times 10^{-9})$ and

$r_w/B = 10^{-3}$, W (from table or from the approximate equation) is equal to 14.05. Thus, $Q = (4\pi)(0.05)(50)/14.05 = 31.42/14.05 = 2.24$ ft³/sec.

For a nonleaky system, Theis formula, Eq. (63), applies since $r_w^2 < \nu t_0/30$. Thus, $Q = 4\pi Ts/W(u)$, or since $u = 1.93 \times 10^{-9} < 0.02$, $Q = 4\pi Ts/\ln(0.562/u) = (31.42)/(2.3)(8.464) = 1.61$ ft³/sec.

Example 12. An artesian sand overlies a thick semipervious layer. The estimated values of T, ν, and $(\sqrt{S'/S})/B$ are, respectively, 0.005 ft²/sec, 50 ft²/sec, and 10^{-3}ft⁻¹. If a steady well is pumped continuously for three days, what percentage of total volume of the water pumped is from leakage? What will be the drawdown at a distance of 4000 ft away from the well at the end of the pumping period if the discharge of the well is 3.14 ft³/sec? What is the corresponding drawdown if the system is assumed nonleaky?

From Eq. (60), the percentage of total volume derived from storage is

$$(V_s/Qt) = 2/\sqrt{\pi nt} - (1/nt)[1 - \exp(nt)\operatorname{erfc}(\sqrt{nt})]$$

$n = (K'/b')S'/S = \nu(S'/SB^2) = (50)(10^{-6})$ sec⁻¹, $t = 3$ days $= 25.92 \times 10^4$ sec; thus, $nt = 12.96$ and $\sqrt{nt} = 3.6$. For $(nt) > 5$, $nt \operatorname{erfc}(\sqrt{nt}) \approx 1/\sqrt{\pi nt} = 0.157$. Thus, $V_s/Qt = 0.314 - 0.064 = 0.25$. Hence, $V_L/Qt = 0.75$.

The drawdown is given by $s = (Q/4\pi T)H(u,\beta)$. For $u = r^2/4\nu t = 0.31 \approx 0.3$ and $\beta = (r/4)(\sqrt{S'/S})/B = 1$, $H(u,\beta)$ is (from Table III) $= 0.173$. Thus, $s = (50)(0.173) = 8.65$ ft.

If the system is assumed nonleaky, then $s = (Q/4\pi t)W(u)$. For $u = 0.3$, $W(u) = 0.905$; hence, $s = (50)(0.905) = 45.25$ ft.

Example 13. A 24-in. flowing well is discharging from a leaky system such as that of Case 1 without storage in semipervious layer. The estimated formation coefficients are $T = 0.0005$ ft²/sec, $S = 9 \times 10^{-5}$, and $K'/b' = 5 \times 10^{-10}$. Estimate the discharge of the well after 0.5, 5, and 50 hr if the constant drawdown is 100 ft. Estimate the time at which the minimum expected discharge will occur and the rate of this discharge. Estimate also the corresponding quantities if the system is not leaky.

The discharge of the well is given by Eq. (70) with $\delta_1 = 1$, or $Q = 2\pi Ts_w G(\tau, r_w/B)$. $\tau = \nu t/r_w^2 = (5 \times 10^{-4}/9 \times 10^{-5})t/1 = 2 \times 10^4 t$ (t in hr). For $t = 0.5$, 5, and 50 hr, $\tau = 10^4, 10^5, 10^6$. For $r_w/B = (1)\sqrt{(K'/b')/T} = 10^{-3}$ and the preceding values of τ, the respective values of $G(\tau, 10^{-3})$ are (from Table II) 0.196, 0.162, and 0.144. Consequently from $Q = 0.314G$, the respective discharges are 0.062, 0.051, 0.045 ft³/sec.

From tables, $G(\tau, 10^{-3})$ is 0.142 for $\tau \geq 3 \times 10^6$. Hence, the minimum discharge is 0.0445 ft³/sec which is attained at approximately $\tau = 3 \times 10^6$, or 150 hours since the start.

For a nonleaky system, $Q = 2\pi Ts_w G(\tau)$. For $\tau = 10^4$, 10^5, and 10^6, $G(\tau)$ is 0.196, 0.161, and 0.136, respectively; hence, Q is 0.062, 0.05, and 0.043 ft³/sec. The minimum discharge is zero.

Example 14. The hydrograph of an observation well, located outside a 10-mi sq irrigation well field, shows that the maximum water levels during shut-down periods have been continuously declining. The corresponding semilogarithmic plot of these maximum water levels, with time plotted on the logarithmic scale, exhibits a straight-line relationship after the first two years since the water levels began to decline; the initial time is taken as the beginning of the irrigation season following the period during which these maximum water levels had been attaining essentially a constant elevation. The slope of the straight portion of the semilogarithmic plot is measured as 100 ft per logarithmic cycle. The estimated values of T and S are, respectively, 0.10 ft²/sec and 4×10^{-4}. Estimate the average rate at which water from storage is being withdrawn if the aquifer is nonleaky.

The well field may be idealized as a large well (of diameter about ten miles) that has been pumping continuously since the beginning of the irrigation season after which the water levels began to decline. The average rate of withdrawal from storage represents the constant discharge of this well. Since the aquifer is nonleaky, Theis' formula applies if $t > 30r_w^2/\nu$. For $r_w \approx 25{,}000$ ft (approximately 5 miles) and $\nu = T/S = 250$ ft²/sec, the value of t is greater than 7.5×10^7 sec, or approximately t > 2.4 yr. Thus, the decline of maximum water level in the observation well after the first 2.5 yr may be represented by the Theis formula, with Q representing the withdrawal from storage. Since the semilogarithmic plot exhibits a straightline relationship, the logarithmic form of Theis formula applies, or $s = (2.3Q/4\pi T)\log_{10}(2.25\nu t/r^2)$ from which $\partial s/\partial \log_{10} t = 2.3Q/4\pi T$. Thus, the slope of the line in the semilogarithmic plot $= 2.3Q/4\pi T = 100$, from which $Q = (4\pi)(0.1)(100)/2.3 = 54.5$ ft³/sec, or the withdrawal from storage is approximately equal to 39,000 acre-ft/yr.

IV. Wells Partially Penetrating Artesian Aquifers

Wells, of which the water-entry section is less than the aquifer they penetrate, are called *partially penetrating wells*. Unlike the flow toward completely penetrating wells where the main flow takes place essentially in planes parallel to the bedding planes of the formation, the flow toward partially penetrating wells is three-dimensional. Consequently, the drawdown observed in partially penetrating wells will depend, among other variables, on the length and space position of the screened portion (water-entry section) of the observation wells, as well as on that of the pumping or flowing well.

Observations have shown as previously noted, that the conductivity along the bedding plane of a formation may be appreciably different from that across the bedding plane. While this anisotropy may not be of great significance in problems where the velocities are essentially confined to planes parallel to the bedding planes, it may be quite important where the flow is three-dimensional, such as that due to partially penetrating wells. In aquifers where the horizontal conductivity is several times greater than the vertical, the yield of partially penetrating wells may be appreciably smaller than that of equivalent wells in isotropic sands [7].

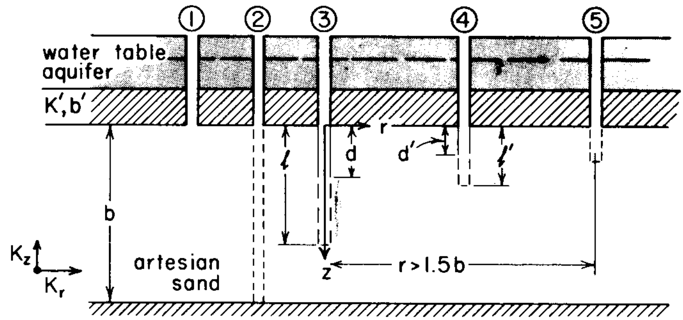

FIG. 10. Diagrammatic representation of wells partially penetrating a leaky artesian aquifer.

In treating the problem of flow toward partially penetrating wells (Fig. 10), the following assumptions are made: (1) The conductivities of the main aquifer in the horizontal and vertical directions have different, but constant values, K_r, K_z, respectively; (2) The slope of the leaky system is so small as to be assumed horizontal, and the system is overlain by a constant-head water body; (3) the storage in the semipervious layer is negligible; and (4) the well is of a vanishingly small radius and discharging at a constant rate.

A. APPROXIMATE EQUATION OF MOTION

The head distribution in each layer of a system involving flow through layered aquifers is a solution of Eq. (14), satisfying among other conditions those at the interfaces (Section I, D, 2, *c*). The drawdown distribution in the semipervious layer of the leaky system of Fig. 10 can be readily obtained in terms of the drawdown in the main aquifer by solving Eqs. (32), Case 1, with $S' = 0$, or $\nu' = \infty$. With the coordinate system of Fig. 10, with z positive downward, this solution is $s_1 = (1 + z/b')\, s$ (at interface), from which the rate of leakage crossing the interface is $w = K'\, \partial s_1/\partial z = (K'/b')\, s$ (at interface). A solution of Eq. (14), with $F = 0$, satisfying the preceding condition (among others), will give solutions to flow problems in the main aquifer. The solution for the flow toward a well partially penetrating the main aquifer thus obtained is complicated and formidable to use in practice. If, instead of being represented as a condition on the boundary, the leakage, actually crossing the interface, is hypothetically assumed to be generated within the main aquifer whose confining beds are now assumed completely impermeable, a solution amenable to relatively easy calculation and sufficiently accurate for practical purposes can be obtained. The rate F of the hypothetically generated leakage per unit volume at each point of the aquifer is taken as $(K'/b')\, s\,(r,t)/b$. This idealization of the flow system does not materially affect the flow pattern obtaining in the actual flow system, since the main aquifer is much more conductive than the semipervious layer. The approximate differential equation governing the flow in this idealized system is that given by Eq. (14), with $F = (K'/b'b)s$. For an anisotropic medium of a constant horizontal conductivity K_r and a vertical conductivity K_z, Eq. (14), in purely radial cylindrical coordinate system and in terms of the drawdown s, where $s = \varphi_i - \varphi$ [φ_i being the initial head distribution; assumedly a solution of Eq. (14)] reduces, with $F = (K'/b'b)s$, to

$$a^2[\partial^2 s/\partial r^2 + (1/r)\, \partial s/\partial r] + \partial^2 s/\partial z^2 - s/B_z^2 = (1/\nu_z)\, \partial s/\partial t \tag{72}$$

where $a^2 = K_r/K_z$, $B_z^2 = K_z b/(K'/b')$, and $\nu_z = K_z b/S$.

B. The Boundary-Value Problem

The flow toward a steadily discharging partially penetrating well screened between penetration depths d and l in the flow system of Fig. 10 is governed by Eq. (72) and the following boundary conditions:

$$s(r,z,0) = s(\infty,z,t) = 0, \qquad \partial s(r,0,t)/\partial z = \partial s(r,b,t)/\partial z = 0,$$

and

$$\begin{aligned} \lim_{r\to 0}\, [(l-d)(r\partial s/\partial r)] &= 0 && 0 < z < d \\ &= -Q/2\pi K_r && d < z < l \\ &= 0 && l < z < b \end{aligned}$$

If, in the above boundary-value problem, the substitution $r = ar'$ is made, the problem changes to

$$\partial^2 s/\partial r'^2 + (1/r')\, \partial s/\partial r' + \partial^2 s/\partial z^2 - s/B_z^2 = (1/\nu_z)\, \partial s/\partial t$$

$$s(r',z,0) = s(\infty,z,t) = 0, \qquad \partial s(r',0,t)/\partial z = \partial s(r',b,t)/\partial z = 0$$

$$\begin{aligned} \lim_{r'\to 0}\, [(l-d)(r'\, \partial s/\partial r')] &= 0 && 0 < z < d \\ &= -Q/2\pi K_r && d < z < l \\ &= 0 && l < z < b \end{aligned}$$

which is of the same form as that of Example 9; that is, if the parameters r,K,B, and ν of Example 9 are replaced by r',K_r,B_z and ν_z, the preceding set of equations results. Thus, the solution to the preceding set of equations is that given by the solution of Example 9, if these replacements are made. The solution to the present problem of a partially penetrating well is then obtained by replacing r' with r/a.

C. Drawdown Equations in Leaky Aquifers

1. General Equation

The drawdown $s(r,z,t)$ in a piezometer having a depth of penetration z and being at a distance r from a steadily discharging well that is screened between the penetration depths d and l in the anisotropic leaky artesian aquifer of Fig. 10 is given by

$$s = (Q/4\pi K_r b)\{W(u_r, r/B_r) + f(u_r, r/b, r/B_r, d/b, l/b, z/b)\} \tag{73}$$

where $u_r = r^2/4\nu_r t, \quad \nu_r = K_r b/S, \quad B_r^2 = K_r b/(K'/b')$, and

$$f = [2b/\pi(l-d)]\sum_{n=1}^{\infty}(1/n)[\sin(n\pi l/b) - \sin(n\pi d/b)] \cdot \cos(n\pi z/b)W(u_r, \sqrt{(r/B_r)^2 + (K_z/K_r)(n\pi r/b)^2}) \tag{74}$$

2. Equation for Relatively Large Time ($t > bS/2K_z$)

For $u < \beta^2/20$, the function $W(u,\beta)$ can, for all practical purposes, be replaced by $2K_0(\beta)$. Thus, for $u_r < (K_z/K_r)(\pi r/b)^2/20$ or for $t > bS/2K_z$, the series of Eq. (74) becomes independent of time, and the solution, or Eq. (73), becomes

$$s = (Q/4\pi K_r b)\{W(u_r, r/B_r) + f_s(r/b, r/B_r, d/b, l/b, z/b)\} \tag{75}$$

with

$$f_s = [4b/\pi(l-d)]\sum_{n=1}^{\infty}(1/n)[\sin(n\pi l/b) - \sin(n\pi d/b)] \cdot \cos(n\pi z/b)K_0(\sqrt{(r/B_r)^2 + (K_z/K_r)(n\pi r/b)^2}) \tag{76}$$

3. Equation for Steady State

This is given by Eq. (75), as $t \to \infty$, or $u \to 0$, that is, the function W is replaced by $2K_0(r/B_r)$.

In practice, the term $(r/B_r)^2$ in the arguments of the functions W and K_0 of Eqs. (74) and (76) is very small compared to $(n\pi r/b)^2(K_z/K_r)$ and, therefore, may be neglected for all practical purposes.

4. Equation for Relatively Large Distances (at $r > 1.5\,b\sqrt{K_r/K_z}$)

For $\beta > 4$, the function $W(u,\beta)$, and hence $K_0(\beta)$, approach zero, and the series of Eqs. (74) and (76) will be of insignificant numerical value relative to that of the function $W(u,r/B)$. Thus, for $(\pi r/b)\sqrt{K_z/K_r} > 4$, or about $r > 1.5b\sqrt{K_r/K_z}$, the series in Eqs. (74) and (76) may be safely neglected. Consequently Eqs. (73) and (75) will both become

$$s = (Q/4\pi K_r b)W(u_r, r/B_r) \tag{77}$$

In fact, Eq. (77) gives results sufficiently accurate for practical purposes even for (r/b) as small as $\sqrt{K_r/K_z}$, provided $u_r < 0.1(r/b)^2(K_z/K_r)$. Equation (77) is the same as it would be if the pumped well completely penetrated the aquifer (Hantush-Jacob formula). In other words, the actual three-dimensional flow pattern changes to a radial type and is hardly distinguishable from that of a purely radial system at a distance from the pumped well equal to or greater than $1.5b\sqrt{K_z/K_r}$. Moreover, the flow will be as if the aquifer were isotropic with a conductivity equal to K_r.

5. *Equations for Average Drawdown*

The water level in an observation well reflects the average head, and consequently the average drawdown, in the aquifer profile that is occupied by the water-entry section of the well. The average drawdown in an observation well screened between the depths d' and l' ($l' > d'$) is obtained by integrating the equation of drawdown in piezometers, Eqs. (73) and (75), with respect to z between the limits d' and l', and then dividing the result by $(l' - d')$. This operation can readily (merely integrating cosine terms) be performed on Eqs. (73) and (75) to obtain the corresponding equations for the average drawdown.

The average drawdown in an observation well screened throughout the aquifer, however close it may be to the partially penetrating pumping well, is given by the Hantush-Jacob formula. This is because $\int_o^b \cos(n\pi z/b)\, dz = 0$; consequently, the series in the drawdown equations drop out after averaging the drawdown along the face of a well that is screened throughout the aquifer. This means that the water levels in such wells are not affected by partially penetrating pumping wells.

6. *Drawdown at the Face of Pumping Well*

The equations of drawdown that have been presented in the previous paragraphs are derived on the assumption that the flux entering the pumped well is uniformly distributed along the water-entry face of the well. Theoretically, the drawdown, rather than the flux, along the face just outside of the pumped-well screen should be uniform. This uniform drawdown distribution can be achieved by a lengthy process involving the distribution of varying flux elements along the well axis. The drawdown at points not in the immediate vicinity of the pumped well, obtained by the simpler equations that are based on a uniform flux across the well screen, does not deviate appreciably from that obtained by the assumedly more exact, but lengthy, process. In the actual problem, neither a uniform flux nor a uniform drawdown is really conceived along the face of the well, because of several involved field and operational conditions. The drawdown, in this case, will have a value between the two theoretical extremes. Thus, each of two theoretical expressions can be used to closely represent the actual problem. The simpler equations are, of course, more appealing.

The theoretical drawdown equations, derived on the assumption of uniform flux along the well screen, will obviously give variable drawdown distribution along the face of the pumped well. It has been shown [7], however, that the maximum drawdown (least hydraulic head) at the face of the well, as given by these equations is closely equal to that which obtains if the drawdown along the face of the well is maintained uniform. The point along the face of the well at which the maximum drawdown occurs depends on the space position of the well screen. If the well is screened throughout its depth of penetration, the maximum drawdown takes place at the top of the aquifer ($z = 0$). If, on the other hand, the well completely penetrates the aquifer and only its lower part is screened, the maximum drawdown occurs at the bottom of the aquifer ($z = b$). But if the well is screened between the depths d and l, the maximum drawdown develops at a depth somewhere between l and d, being closer to d if most of the well screen is in the upper half of the aquifer, and closer to l if the reverse is true. Computation in the drawdown equation shows, however, that the value of maximum drawdown does not differ appreciably from that obtained for $z = 0.5(l+d)$. Consequently, the water level at the face of a pumping well (equal to the drawdown in the well only if well losses are neglected) can be calculated from the equations of drawdown in piezometers, Eqs. (73) and (75), by substituting therein r_w for r, and the value of z at which the maximum drawdown takes place.

Most frequently the pumping well is screened throughout its depth of penetration. In this instance, the equation for the drawdown at the face of the well s_w is considerably simplified. For $t > Sb/2K_z$, this equation will be given by Eq. (75), with $r = r_w$, $z = 0$, and $r_w/B_r \approx 0$. An equivalent equation, suitable for computation when $(r_w/b)\sqrt{K_z/K_r}$ is small, can be obtained after mathematical transformation [7] and neglecting terms of second and higher powers of $(r_w/b)\sqrt{K_z/K_r}$, as:

$$s_w = (Q/4\pi K_r b)\{W(r_w^2/4v_r t,\ r_w/B_r) + (2b/l) \cdot [\sinh^{-1}[(l/r_w)\sqrt{K_r/K_z}] - (l/b)\ln[(4b/r_w)\sqrt{K_r/K_z}] - \ln[\Gamma(1 + l/2b)/\Gamma(1 - l/2b)]]\}$$

in which $\Gamma(x)$ is the gamma function and $\sinh^{-1}(x)$ is the inverse hyperbolic sine, tabular values for both functions are available [15].

For $(l/r_w)\sqrt{K_r/K_z} > 10$, which generally obtains in practice, and $0 \leq l/b \leq 0.5$, the equation may, from Section II, C, 14 and 20, be written as

$$s_w \approx (Q/4\pi K_r b)[W(r_w^2/4v_r t,\ r_w/B_r) + F] \qquad (78)$$

where

$$F = (2b/l)\{(1 - l/b)\ln[(2l/r_w)\sqrt{K_r/K_z}] - (l/b)\ln(2b/l) - (0.423l/b) + \ln[(2b + l)/(2b - l)]\}$$

D. Drawdown Equations in Nonleaky Aquifers

The equations of drawdown, when leakage is neglected, are obtained from their counterparts for leaky aquifers by making K' equal to zero, or $B_r \to \infty$.

1. General Equation

From Eq. (73), with $B_r = \infty$, the required equation is

$$s = (Q/4\pi K_r b)\{W(u_r) + f(\text{with } r/B_r = 0)\} \tag{79}$$

An equivalent equation suitable for computation when r/b is small is available [21].

2. Equation for Relatively Large Time ($t > bS/2K_z$)

From Eq. (75), with $B_r = \infty$, the required equation is

$$s = (Q/4\pi K_r b)\{W(u_r) + f_s(\text{with } r/B_r = 0)\} \tag{80}$$

which for $(r/b)\sqrt{K_z/K_r} < 0.1$, may, for practical purposes, be approximated by

$$s = (Q/4\pi K_r b)[W(u_r) + f_s'] \tag{81}$$

in which

$$\begin{aligned} f_s' = {} & [b/(l-d)]\Big\{\sinh^{-1}[a(l+z)/r] + \sinh^{-1}[a(l-z)/r] \\ & -(2l/b)\ln(4ba/r) + \ln\frac{\Gamma[1-(l+z)/2b]\,\Gamma[1-(l-z)/2b]}{\Gamma[1+(l+z)/2b]\,\Gamma[1+(l-z)/2b]}\Big\} \\ & - [b/(l-d)]\{\text{same braced terms with } d \text{ replacing } l\} \end{aligned}$$

and where

$$a = \sqrt{K_r/K_z}$$

3. Equation for Short Time ($t < Sb[1-(l+z)/2b]^2/5K_z$)

In this range of time, it has been shown [21] that the drawdown in piezometers may be approximated by

$$s = [Q/8\pi K_r(l-d)]E(u_r, l/r, d/r, z/r) \tag{82}$$

where

$$\begin{aligned} E = {} & M[u_r, a(l+z)/r] - M[u_r, a(d+z)/r] \\ & + M[u_r, a(l-z)/r] - M[u_r, a(d-z)/r] \end{aligned}$$

and $M(u,\beta)$ is an infinite integral (Section II, C, 12), tabular values of which are given in Table IV.

Equation (82) also gives the drawdown in an infinitely deep aquifer for the whole range of time.

4. Equation for Relatively Large Distances (at $r > 1.5b\sqrt{K_r/K_z}$)

As $B \to \infty$, Eq. (77) reduces to the Theis formula; namely,

$$s = (Q/4\pi K_r b)W(u_r) \tag{83}$$

5. Equations for Average Drawdown

The average drawdown in an observation well screened between the depths l' and d' is readily obtained by integrating Eqs. (79) and (80) with respect to z between the limits d' and l', and then dividing the result by $l' - d'$. However, computations in these equations [21] show that if $l'/l < 2$, results sufficiently accurate for practical application can be obtained from Eqs. (81) and (82), each within its range of time, if z is replaced by $0.5(l' + d')$. In other words, the average drawdown in an observation well screened between the depths l' and d' can be approximated by the drawdown registered in a piezometer whose penetration depth is $0.5(l' + d')$, provided that $l'/l < 2$.

Also, if $(r/l)\sqrt{K_z/K_r} > 1$ and $l'/l < 1$, the average drawdown in the observation well can, for all practical purposes, be taken as that given by Eqs. (81) and (82), each within its own limitation, the value of z being arbitrarily chosen between l' and zero. The choice is generally made in such a manner as to simplify the equation.

6. Drawdown at the Face of Pumping Well

The equations for this drawdown are obtained from their counterparts for leaky aquifers, with $B = \infty$.

For $t > Sb/2K_z$, the equation of drawdown is given by Eq. (78) or the one just preceding it, after replacing $W(u,\beta)$ by $W(u)$, provided the well is screened throughout its depth of penetration.

For $t < Sb(1 - l/2b)^2/5K_z$ or if the aquifer is infinitely deep, the drawdown at the face of a pumping well screened throughout its depth of penetration is, from Eq. (82) with $d = 0$ and $z = 0$, given by

$$s = (Q/4\pi K_r l)M(r_w^2/4\nu_r t,\ al/r_w) \tag{84}$$

E. Recovery Equations

The recovery equation corresponding to any of the drawdown equations presented in this section can be written readily through use of Eq. (65) subject to the same time criteria.

F. General Remarks and Examples

The effect of partial penetration on the drawdown around a pumping well is shown in Figs. 11a and 11b. The variations are around a well in an isotropic aquifer ($K_r = K_z$).

The average drawdown developed in an observation well, however close it may be to the partially penetrating well, is given by the Hantush-Jacob formula for leaky aquifers, Eq. (77), or by the Theis formula for nonleaky aquifers [Eq. (83), with $K = K_r$], provided that the observation well is screened throughout the aquifer. The same is true for a well located at $r > 1.5b\sqrt{K_r/K_z}$, regardless of the space position of its screen. In other words, the average drawdown in such wells is not affected by partial penetration; it is the same as though the pumped well completely penetrated the aquifer (curve 1 of Fig. 11a).

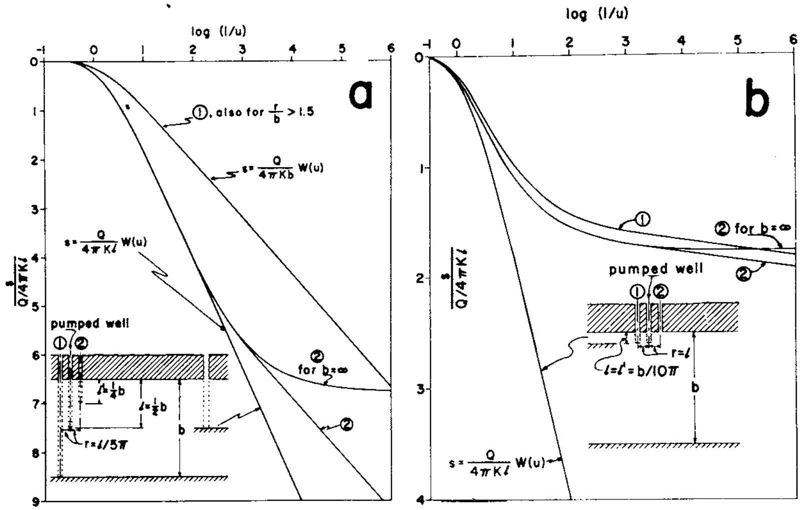

FIG. 11. Time-drawdown variation in wells partially penetrating a nonleaky artesian aquifer.

Regardless of the location of the wells and the space position of their screens, the time-drawdown curves, at relatively large values of time ($t > Sb/2K_z$), will have approximately the same slope. This slope is the same as would obtain if the pumped well completely penetrated the aquifer. In other words, the effect of partial penetration has attained its maximum value (curves 1 and 2 of Fig. 11a).

If the observation well is not relatively distant ($r < 1.5b\sqrt{K_r/K_z}$) or is not screened throughout the aquifer, the variation of the average drawdown with the logarithm of time will have the trend shown by the curve labeled "2" in Fig. 11a and curves 1 and 2 of Fig. 11b. During the early period of pumping and before the inflection of the curves appears, such time-drawdown curves have the same general appearance as curves for the complete-penetration cases (Hantush-Jacob and Theis formulas); these formulas, of course, are inapplicable. Even in the very early period of pumping, when one may be tempted (for the purpose of applying the simple formulas) to assume that the aquifer ends at the bottom of the pumped well, these formulas cannot be applied except when the geometry of the flow system is such that the drawdown equation for relatively short time reduces to the form of Eq. (84), in which case the validity of using the Theis formula is assured only in the range where $t < (l/b)^2 bS/20K_z$.

Figure 11b compares the drawdowns observed in two equally distant wells, one of zero penetration and the other screened throughout its depth of penetration. It shows that two wells equally distant from a partially penetrating pumping well may register two different drawdowns. In fact, depending on the length and the relative positions of the screens, it is possible for a more distant well to reflect a greater drawdown.

The effects of partial penetration resemble the effects of leakage from storage in a thick semipervious layer (Fig. 6). Also, if the curve inflection is apparent, but the period of observation is not long enough to establish the ultimate straight-line variation on a semilogarithmic time-drawdown plot, the effects of partial penetration resemble the effects of some kind of recharge boundary, such as induced infiltration from streams or lakes (Fig. 19), or recharge from water-bearing strata supplying leakage through semipervious confining beds (Fig. 8). The same general effects are observed if the wells completely penetrate a sloping water-table aquifer (Fig. 17) or an aquifer of nonuniform thickness [29]. Thus, without sufficient information about a flow system that is being studied, observational drawdown trends may be interpreted in several ways. Indiscriminate use of such data may give erroneous and, in many instances, unreasonable results.

A partially penetrating well will discharge less than a completely penetrating well if the two are operated at the same pumping level, other conditions controlling the flow remaining constant. If they are pumped at the same rate, however, the pumping level of the former will be lower than that of the latter.

Pumping at the same level, the yield of a partially penetrating well in an anisotropic aquifer ($K_r \neq K_z$) will decrease with decreasing K_z/K_r, other conditions being the same. The effect of the anisotropy decreases as the well penetration increases. If K_z/K_r does not differ greatly from unity, the anisotropy will not be of particular consequence except for very small penetration. On the other hand, should K_z/K_r be very small, the anisotropy of the aquifer may cause an appreciable decrease in the yield of the partially penetrating well. If K_z should actually vanish, the flow toward the well will become purely radial, confined to the part of the aquifer in which the well is screened.

Example 15. An artesian well screened throughout its 50-ft depth of penetration is pumped and by observing water levels in wells more than 1000 feet away, the values of T and S of an assumedly isotropic aquifer are found as 0.10 ft²/sec. and 4×10^{-3}, respectively. On the assumption that the aquifer thickness is 50 ft, an estimate is made for the yield of a 24-in. well to be screened throughout its 50-ft depth of penetration. The estimate is made such that the drawdown at the face of the well (equal to drawdown in the well, if well losses are neglected) is not to exceed 30 ft after about 3 days (25×10^4 sec) of continuous pumping. The aquifer is later found to be 200 ft thick. Determine the error in the estimate of the well yield.

On the assumption that the well completely penetrates the aquifer, the yield, from the Theis formula, is $Q = 4\pi Ts/W(u)$. For $u = r_w^2 S/4Tt = (1)^2(4 \times 10^{-3})/(4)(0.1)(25 \times 10^4) = 4 \times 10^{-8}$, $W(u) \approx 16.46$; so $Q = (4\pi)(0.1)(30)/16.46 \approx 2.3$ ft³/sec.

For a partially penetrating well, Eq. (78) applies if $t > Sb/2K_z$ and $(l/r_w)\sqrt{K_z/K_r} > 10$. Since $K_z = K_r = T/b = 0.1/200$, then $t = 25 \times 10^4 > (4 \times 10^{-3})(200)/2(0.0005) = 800$ sec, and $l/r_w = (50/1) > 10$, Eq. (78) applies. Thus, for $W(u,0) = W(u) = 16.46$ and $F = 24.6$, Eq. (78) gives $Q = 4\pi Ts/[W(u) + F] \approx 0.91$ ft³/sec. The estimated yield is, therefore, about 150% more than the actual yield.

Example 16. If the aquifer of Example 15 is later found to be anisotropic and having $K_r = 9K_z$, determine the error in the first and second estimates of the yield.

The determined value of T corresponds to the horizontal conductivity, or $T = K_r b$. Equation (78) still applies since for $K_z/K_r = 9$, the criterion $t > Sb/2K_z = 7200$ sec, and the criterion $10 < (l/r_w)\sqrt{K_z/K_r} = 16.67$, still apply. For $K_r/K_z = 9$, the value of F of Eq. (78) is ≈ 31.3. Consequently, $Q = 4\pi Ts/[W(u) + 31.3] \approx 0.80$ ft³/sec. Thus, in the first two estimates the well yield is overestimated by approximately 190 and 11.4%, respectively.

Example 17. The estimated values of S, K_r, and K_z are 5×10^{-3}, 10^{-4} ft/sec, and $(1/9)K_r$, respectively. Estimate the yield of a 24-in. well screened throughout its 100-ft depth of penetration if the drawdown at the face of the well is not to exceed 50 ft after about 7 hr (2.5×10^4 sec) of continuous pumping from the 500-ft thick aquifer. Estimate the expected drawdowns in two observation wells, one being 100 ft from the pumping well and just tapping the aquifer (approximately zero penetration, or $z = 0$), and the other being 50 ft away and screened throughout its 200-ft depth of penetration. What would be the corresponding yield of the well if the aquifer were assumed isotropic with K_r equal to the estimated value of $K_z = 1/9 \times 10^{-4}$ ft/sec?

Since $t = 2.5 \times 10^4 < bS(1 - l/2b)^2/5K_z = (500)\ (5 \times 10^{-3})[1 - 100/2(500)]^2/5(1/9)10^{-4} = 3.5 \times 10^4$ sec, Eq. (84) applies. Thus, $Q = 4\pi K_r ls/M(u, \beta)$. For $u = (r_w^2 S/4K_r bt) = (1)(5 \times 10^{-3})/(4)(10^{-4})(500)(2.5 \times 10^4) = 10^{-6}$, and $\beta = (l/r_w)\sqrt{K_r/K_z} = (100/1)(3) = 300$, M (from Table IV) ≈ 12.12, so $Q = 4\pi(10^{-4})(100)(50)/(12.12) \approx 0.52$ ft³/sec.

For observation wells 1 and 2, Eq. (82) (with $d = 0$, $z = 0$, and $r = 100$ ft) gives $s_1 = (Q/4\pi K_r l)M(u_1, \beta_1)$; with $d = 0$, $r = 50$ ft, and since $(l'/l) = 200/100 \leq 2$, $z = 0.5\,(d' + l') = 0.5\,(0 + 200) = 100$ ft, Eq. (82) gives $s_2 = (Q/8\pi K_r l)\,M(u_2, \beta_2)$, where the subscripts 1 and 2 pertain to wells 1 and 2, respectively. For $u_1 = u(r_1/r_w)^2 = 10^{-2}$ and $\beta_1 = \beta(r_w/r_1) = 3$, the value of $M(u_1, \beta_1) = 2.97$; thus, $s_1 = s\,M(u_1, \beta_1)/M(u, \beta) = (50)(2.97)/(12.12) = 12.3$ ft. For $u_2 = u(r_2/r_w)^2 = 2.5 \times 10^{-3}$ and $\beta_2 = \beta(r_w/r_2) = 6$, the value of $M(u_2, \beta_2) = 4.32$; thus, $s_2 = s\,M(u_2, \beta_2)/2M(u, \beta) = (50)(4.32)/(2)(12.12) = 8.9$ ft. For an isotropic aquifer with $K_i = (1/9) \times 10^{-4}$, the yield will be $Q_i = 4\pi K_i ls/M(u_i, \beta_i)$, with $u_i = 9u = 9 \times 10^{-6}$ and $\beta_i = (1/3)\beta = 100$, hence $M(u_i, \beta_i) = 9.93$. Thus, $Q_i = (K_i/K_r)(M/M_i)Q = (1/9)(12.12/9.93)(0.52) = 0.07$ ft³/sec. The subscript i pertains to the isotropic aquifer.

Example 18. Show that, if the vertical conductivity K_z of an aquifer becomes effectively zero, the flow around a partially penetrating well will become purely radial, confined to the part of the aquifer in which the well is screened.

For $K_z = 0$, Eq. (82) applies for the whole range of time ($t <$ constant/(0) $= \infty$). Thus, from Eq. (82), the drawdown at points $(r, z < d)$ is $(s/A) = M(u_r, \infty) - M(u_r, \infty) + M(u_r, \infty) - M(u_r, \infty) = 0$, and at points $(r, z > l)$, $(s/A) = M(u_r, \infty) - M(u_r, \infty) + M(u_r - \infty) - M(u_r, -\infty) = 0$. But at points $(r, d < z < l)$, $(s/A) = M(u_r, \infty) - M(u_r, \infty) + M(u_r, \infty) - M(u_r, -\infty) = 2M(u_r, \infty) = 2W(u_r)$. Hence, $s = [Q/4\pi K_r(l-d)]\,W(u_r)$, which is the equation for purely radial flow (Theis equation) in an aquifer of thickness equal to $l-d$.

V. Flow to Water-Table Wells

Wells tapping water-table aquifers are called *water-table wells.* They are also known as *gravity wells.* The flow of ground water to such wells, or in water-table aquifers in general, is extremely complicated because the position of the upper boundary of the flow, the water table, is not known. A further complication arises from the occurrence in such flow systems of the so-called seepage faces, from which ground water emerges either to evaporate or trickle down the boundary face. The length of these faces is also unknown, as their upper terminal joins with the water table which approaches the seepage face tangentially. The existence of these faces has been demonstrated experimentally, rationally, and analytically by several investigators [7, 30, 31].

Exact solutions for problems in gravity-flow systems (water-table systems) are solutions of Eq. (15), satisfying, among other conditions, the boundary conditions on the water table and the seepage faces (Section I, D, 2, *a* and *d*). In general, it is extremely difficult, if not impossible, to satisfy analytically these boundary conditions. The length of the seepage face depends on the position of the water table which is unknown. The shape of the water table determines the distribution of flow underneath, whereas its shape in turn depends upon that distribution. The hodograph method developed by Hamel and Muskat [7] does, in principle, provide a means for treating almost any two-dimensional system, but the labor of applying it numerically is so formidable that it is not suitable in practical applications. Accordingly, certain approximations will be made such that flow systems of general interest may be described by conditions and differential equations that are amenable to relatively easy solutions with results that are sufficiently accurate for practical use.

A. Steady-State Flow to Wells in Horizontal Nonleaky Aquifers

When the discharge of a well is supplied from storage only, as is the case in an infinite horizontal nonleaky aquifer, steady-state flow will theoretically never be attained. It other words, the flow around the well is always transient, and the zone of influence is always expanding. The flow appears, however, to approach a steady state with time because the decline of water levels around the well decreases in magnitude with both time and distance from the well. Essential stability of flow in the immediate vicinity of pumping wells may be reached within a relatively short time (depending on the hydraulic properties of the aquifer) after pumping begins. The area of essential stability expands continuously and a considerable period of time may be required for the water levels in areas far from the pumped wells to reach approximate equilibrium. Within the area of essential stability where no appreciable changes of water levels occur as a result of pumping, the flow toward the well may be treated as though it were in dynamic equilibrium (steady state) with a hypothetical body of water supplying the total discharge of the well and maintaining the original depth of flow. The hypothetical body of water, concentric with the well, is located at a hypothetical distance r_e outside the area of essential stability such that the actual shape of the water table within the area of essential stability will be reproduced (outside this area the actual water table may be quite different from that reproduced by this hypothetical system). As the area of essential stability expands, the value of r_e will expand, although very slowly, to reproduce the actual water table in the newly stabilized region; within the original area of essential stability, however, the shape of the water table remains essentially the same, and the original value of r_e need not be changed within a period of continuous pumping that will not materially change the water levels within this region. The hypothetical body of water may be regarded as the source of recharge sustaining the well discharge most of which is actually derived from storage outside the area of essential stability, where the dewatering process is continuing.

1. Equations for the Water Table

a. EXACT EQUATION. A gravity radial flow system is shown diagrammatically in Fig. 12. The isotropic and homogeneous aquifer, resting on a horizontal, impermeable bed, is being drained by a well that is screened throughout and that completely penetrates the aquifer. The flow is under dynamic equilibrium.

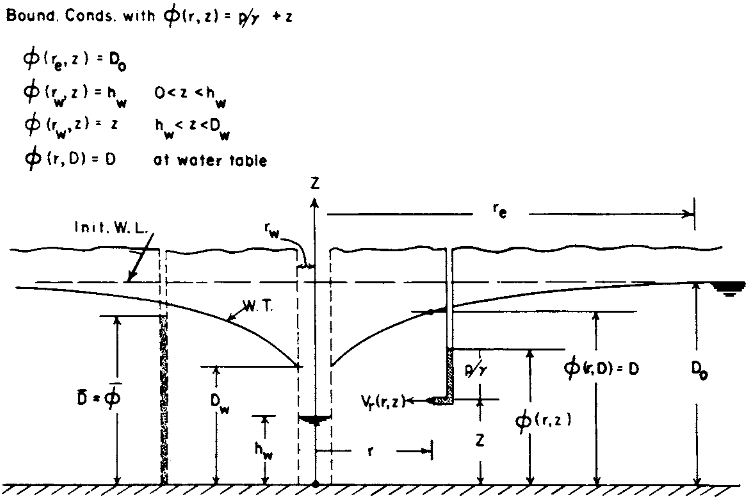

Fig. 12. Diagrammatic representation of flow to a well in a horizontal water-table aquifer.

The depth of saturation at the face of the well is D_w, and the depth of water in the well is h_w. The well losses are zero. In the region of undisturbed flow, the depth of saturation is D_0. If the height of the water table above the base of the aquifer at the radial distance r from the center of the well is designated by D, and if r_e and r_w are the radius of the outer boundary (the hypothetical body of water) and the radius of the well, respectively, the boundary-value problem describing the flow in such a system is given by

$$v_r(r,z) = -K\,\partial\varphi(r,z)/\partial r \tag{a}$$

$$\varphi(r,z) = p/\gamma + z \tag{b}$$

$$\varphi(r,D) = D(r) \tag{c}$$

$$\varphi(r_e,z) = D(r_e) = D_0 \tag{d}$$

$$\partial\varphi(r,0)/\partial z = 0 \tag{e}$$

and

$$\varphi(r_w,z) = h_w \quad \text{for} \quad 0 < z < h_w,$$

and

$$\varphi(r_w,z) = z \quad \text{for} \quad h_w < z < D_w \tag{f}$$

where $v_r(r,z)$ is the bulk velocity at any point (r,z) taken positive in the direction of r; other symbols have been defined.

Consider a cylindrical prism of the aquifer of radius r concentric with the well. As the water table is a stream surface and the base of the aquifer is impermeable (condition e), and since the flow is in the steady state, the total net inward flow across the cylindrical surface must equal the discharge of the well. Thus,

$$Q = -2\pi r \int_0^{D(r)} v_r(r,z)\,dz = 2\pi r K \int_0^{D(r)} [\partial\varphi(r,z)/\partial r]\,dz$$

Through use of the rule of differentiation under the integral sign, the preceding equation may be put in the form

$$Q/2\pi Kr = -\varphi(r,D)\,dD/dr + d\Big[\int_0^{D(r)} \varphi(r,z)\,dz\Big]/dr$$

When $\varphi(r,D)$ is replaced by its equivalent from condition (c) and after terms are collected, the above relation reduces to

$$d\Big[-D^2 + 2\int_0^{D(r)} \varphi(r,z)dz)\Big]/dr = Q/\pi Kr$$

or, after integrating with respect to r between the limits r and r_e (the corresponding limits for D are D and D_0) and making use of condition (d), it becomes

$$2D\bar{D} - D^2 = D_0^2 - (Q/\pi K)\ln(r_e/r) \qquad (85)$$

where $\bar{D} = \bar{\varphi} = (1/D)\int_0^{D(r)} \varphi(r,z)\,dz$, which is the depth of water in an observation well that is screened throughout the aquifer, or the average hydraulic head intercepted by this observation well.

Equation (85) is useful for calculating the exact position of the water table around a gravity well only if observational data from nearby completely penetrating wells are available; other formation parameters are assumedly known.

b. DUPUIT-FORCHHEIMER EQUATION. Experimental [30] and theoretical [32] investigations have shown that the difference between D and $\bar{D}$ is not measurable at points for which $r > 1.5D$, even if the depth of the water in the well (not at the face of the well) is zero. Moreover, this difference is still insignificant for smaller r as the water level in the well is increased. Thus in the region where $r > 1.5D_0$ (as D is unknown except when obtained by observation, this criterion is used instead of $r > 1.5D$; sufficiently accurate results for practical applications may be obtained if this criterion is replaced by $r > 1.5D_w$), the water table may be described by Eq. (85) after replacing $\bar{D}$ with D; namely, by

$$D^2 = D_0^2 - (Q/\pi K)\ln(r_e/r) \qquad (86)$$

which is the well-known equation of Dupuit and Forchheimer. As anticipated by theory, experiments [7, 30] and numerical analysis [32] have shown that although Eq. (86) closely represents the water table at $r > 1.5D$, it fails to represent the free surface near and to some distance around the well. The same investigations show that it approximates the head distribution along the base of the aquifer rather than the water table. In fact, when Eq. (86) is used to compute for D at r_w, the value thus computed will be the depth of the water in the well h_w (well losses are neglected) rather than D_w (the height of the water table at the face of the well). Thus another form of Eq. (86) may be obtained, as

$$D^2 = D_0^2 - (D_0^2 - h_w^2)[\ln(r_e/r)/\ln(r_e/r_w)]$$

c. APPROXIMATE EQUATION FOR $r < 1.5D_0$. Within $r < 4r_w$, the shape of the water table may be approximated [33] by

$$D^2 = D_0^2 - (D_0^2 - D_w^2)[\ln(r_e/r)/\ln(r_e/r_w)] \quad (87)$$

for which an estimate of D_w should be available (see Subsection 4, below).

For $4r_w < r < 1.5D_0$, the shape of the water table may be obtained [33], with an overestimating error not exceeding 3.5% from the relation $D = D_1 + D_2$ where D_1 and D_2 are the values of D as computed by Eqs. (86) and (87), respectively. When computing in Eqs. (85) to (87), the values of D_0 and r_e may be replaced by any point (D,r) provided the point is taken in the region $r > 1.5D_0$ (as noted previously, sufficiently accurate results may be obtained for $r > 1.5D_w$).

2. *Well-Discharge Formula*

The average head at the face of the well is

$$\bar{D}(r_w) = (1/D_w)\int_0^{D_w} \varphi(r_w,z)\, dz$$

which from condition (f) of Subsection 1, *a* above, is evaluated as

$$D_w\bar{D}(r_w) = \int_0^{h_w} h_w\, dz + \int_{h_w}^{D_w} z\, dz = 0.5(h_w^2 + D_w^2)$$

Thus, at the face of the well, Eq. (85), after solving for Q, gives

$$Q = \pi K(D_0^2 - h_w^2)/\ln(r_e/r_w) \quad (88)$$

This is the Dupuit-Forchheimer well-discharge formula. The formula was originally obtained by assuming purely horizontal flow and ignoring the existence of the seepage face at the well surface. Despite the shortcomings of these assumptions, observations have shown that this equation predicts the discharge of the well with a very high degree of accuracy commensurate with experimental errors. Investigations [7, 30, 32] concluded that this equation gives the correct discharge within 1 to 2%.

In the present derivation, the curvilinear nature of the flow as well as the existence of the seepage face are taken into consideration. Thus, the validity of this very well-known formula is rigorously established. Other analytical procedures also have proved the validity of this equation [34].

In the theoretical development of Eqs. (85) to (88), it is assumed implicitly that D is not equal to $\bar{D}$ except at $r = r_e$. Practically, however, $D = \bar{D}$ for all values of $r > 1.5D$. In other words, the condition $\varphi(r_e,z) = D(r_e) = D_0$, used in obtaining Eqs. (85) to (88) may be replaced with the condition $\varphi(r_i,z) = D(r_i) = D_i$, if $r_i > 1.5D$. Thus the value of D_0 and the usually unknown value of r_e may be replaced by D_i and r_i in Eqs. (85) to (88) with practically no measurable error; r_i being known (selected), D_i is computed from Eq. (88), since r_w, h_w, and Q are known also. If obtained from field observations, the point (D_i,r_i) must be taken within the area of essential stability.

3. The Zone of Influence

As the flow around a well in an assumedly infinite aquifer with no source of recharge is always transient, the zone of influence is always expanding, although the rate of its expansion may be slow after a relatively long period of pumping. Although Eqs. (85) to (88) may be used with an estimated or computed value of r_e to establish the shape of the water table within an area of essential stability, they cannot be used to calculate the actual radius of influence. The actual extent of the zone of influence may be several times greater than the value of r_e as computed by these equations. The true radius of influence may be obtained, however, by using the unsteady equation of drawdown. At any time after pumping begins, the extent of the influence may be taken as that distance r which makes the calculated drawdown in Eqs. (90), (92), and (94) effectively zero, or of the order of 0.01 ft (see Example 19); appropriate drawdown equations for wells in artesian systems may similarly be used. The shape of the water table at any time t within a region defined by $u < 0.05$; that is, $r < 0.45\sqrt{\nu t}$, may, however, be obtained through use of Eqs. (85) to (88) with a value of r_e (the hypothetical radius of influence) equal to $1.5\sqrt{\nu t}$ [see special case of Eq. (92)]. The shape of the water table at this time and in the region $r > 0.45\sqrt{\nu t}$ is, of course, not represented by Eqs. (85) to (88).

4. Height of Seepage Face

From several relaxation and experimental solutions for problems of gravity flow to wells, Boulton [32] concluded that the ratio $2\pi KD_0(D_0 - D_w)/Q$ varies only slightly with r_w/D_0, and that for wells of usual diameters, the height of the seepage face may be obtained, if $r_w/D_0 < 0.1$, from

$$D_w - h_w \approx (D_0 - h_w) - 3.75Q/2\pi KD_0$$

Furthermore, if (r_w/D_0) is of the order of 0.25, the above relation may be used, provided the factor 3.75 is replaced by 3.5.

Based on relaxation and experimental solutions, Hall [35] proposed the following relation

$$D_w - h_w \approx \frac{[D_i - h_w]}{[1 + 5r_w/D_i]} \cdot \frac{[1 - (h_w/D_i)^{2.4}]}{[1 + 0.02\ln(r_i/r_w)]} \qquad (89)$$

in which the symbols have been defined.

Other procedures for estimating the value of D_w are available [36].

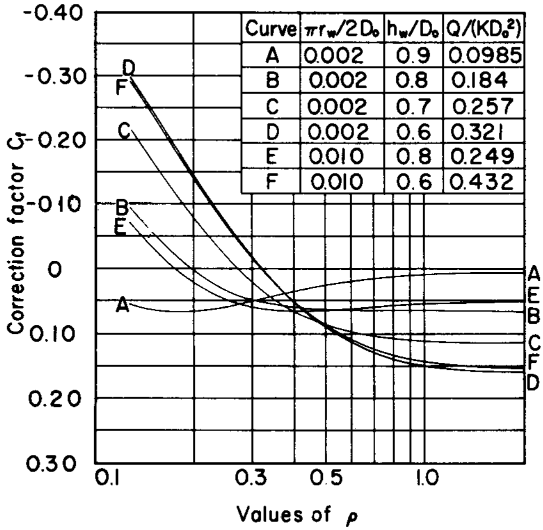

FIG. 13. Boulton's correction factor C_f for $\tau < 0.05$.

B. Unsteady Flow to Wells in Horizontal Nonleaky Aquifers

1. Boulton's Gravity-Well Formula

If, in the flow system of Fig. 12, r_e becomes infinite, the unsteady flow around the well may be described by the following boundary-value problem (see Section, I, D, 2, *d*, for free-surface differential equation):

$$\partial^2\varphi/\partial r^2 + (1/r)\,\partial\varphi/\partial r + \partial^2\varphi/\partial z^2 = (S_s/K)\,\partial\varphi/\partial t \tag{a}$$

$$(\epsilon/K)\,\partial\varphi(r,D,t)/\partial t = [\partial\varphi(r,D,t)/\partial r]^2 + [\partial\varphi(r,D,t)/\partial z]^2 - \partial\varphi(r,D,t)/\partial z \tag{b}$$

$$\varphi(r,z,0) = D_0 \tag{c}$$

$$\varphi(\infty,z,t) = D_0 \tag{d}$$

$$\partial\varphi(r,0,t)/\partial z = 0 \tag{e}$$

$$Q = 2\pi r_w K \int_0^{D_w(t)} [\partial\varphi(r_w,z,t)/\partial r]\,dz \tag{f}$$

A solution for this problem cannot easily be found. Boulton [24] proposed that if $D_0 - h_w < 0.5D_0$, condition (b) may be approximated by $(\epsilon/K)\,\partial\varphi(r, D_0,t)/\partial t = -\partial\varphi(r,D_0,t)/\partial z$; that is the lowering of the water table may be replaced by changes in the head, φ, along the static water-table position where radial flow components may be neglected. Also and under the same condition, condition (f) may be approximated by $Q = 2\pi K r D_0\,\partial\varphi/\partial r$ as $r \to 0$; that is, the constant discharge of the well, instead of being received along the depth of saturation at the face of the well, is received along the original depth

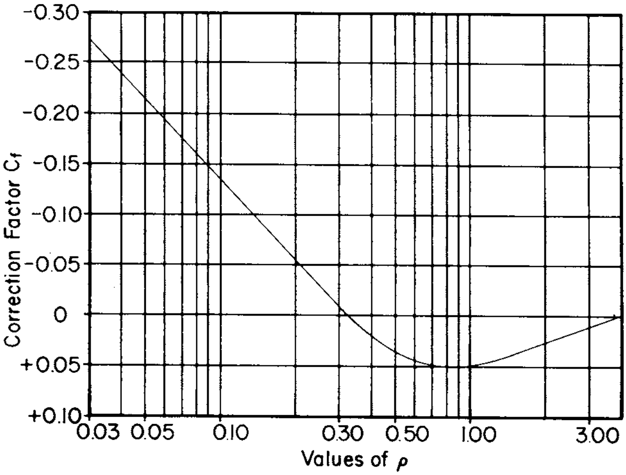

FIG. 14. Boulton's correction factor C_f for $\tau > 5$.

of saturation. Furthermore, in as much as the value of specific storage S_s for water-table aquifers is very small relative to its specific yield ϵ and, therefore, the contribution to the flow from aquifer compression may be neglected except during the very early period of induced flow, when the discharge of the drainage facility is almost entirely from the latter source (depending on the character of the formation, this initial period rarely exceeds several scores of minutes), the right-hand side of condition (a) may be assumed zero. Under these conditions, a solution of the boundary-value problem can be obtained (using the Laplace transformation, the finite Fourier cosine transformation, and the Hankel transformation), which, when evaluated at $z = 0$, may be written as

$$s = D_0 - D = (Q/2\pi K D_0) V(Kt/\epsilon D_0, r/D_0)$$

a. EQUATION OF WATER-TABLE DRAWDOWN. By applying a correction factor to minimize the errors involved in obtaining the preceding equation, the drawdown of the water table (which is not equal to the drawdown in observation wells unless $r > 1.5D$) around a steady gravity well completely penetrating a horizontal nonleaky aquifer, as proposed by Boulton, and provided $D - h_w < 0.5\ D_0$ is given by

$$s = (Q/2\pi T_0)(1 + C_f) V(\tau, \rho) \tag{90}$$

in which $\tau = Kt/\epsilon D_0$, $\rho = r/D_0$, $T_0 = KD_0$, and C_f is a correction factor depending on all or part of the parameters τ, ρ, r_w/D_0, and Q/D_0^2. The value of C_f which ranges from about -0.30 to about 0.16 may be obtained from Fig. 13 for $\tau < 0.05$, and from Fig. 14 for $\tau > 5$, and may be assumed zero for $0.05 < \tau < 5$, with an error not exceeding 6%.

If the aquifer is anisotropic with vertical and horizontal conductivities K_z and K_r, respectively, the solution is

$$s = (Q/2\pi K_r D_0)(1 + C_f)V(\tau',\rho')$$

where $\tau' = K_z t/\epsilon D_0$ and $\rho' = (r/D_0)\sqrt{K_z/K_r}$.

For tables and useful approximations of the function V, the reader is referred to Section II, C.

b. EQUATION OF DRAWDOWN IN THE PUMPED WELL. If well losses are neglected, the drawdown in the pumping well may be approximated [24] as follows:

(*i*) *For* $\tau < 0.05$. The drawdown in the well is given by Eq. (90) after replacing D and ρ by h_w and $\rho_w(=r_w/D_0)$, respectively.

(*ii*) *For* $0.05 < \tau < 5$. The drawdown may be computed by

$$D_0 - h_w = (Q/2\pi T_0)[m + \ln(D_0/r_w)]$$

where m is obtained from a curve constructed from the following tabulation:

$\tau =$	5.0	1.0	0.2	0.05
$m =$	1.288	0.512	0.087	−0.043

(*iii*) *For* $\tau > 5$. The required expression is

$$D_0^2 - h_w^2 = (Q/\pi K)\ln(1.5\sqrt{\tau}/\rho_w) \tag{91}$$

By replacing τ and ρ with τ' and ρ' in the above expressions, the resulting expressions will be those for an anisotropic aquifer.

2. *Average Drawdown in Observation Wells*

The water level in an observation well reflects the average hydraulic head in the aquifer profile that is occupied by the screened portion of the well. Since for $r > 1.5D$, the hydraulic head is practically (Section V, A, 1, *b*) independent of the vertical dimension, the average drawdown in observation wells located in the region $r > 1.5D_0$ is practically equal to the drawdown of the water table and, therefore, is given by Eq. (90).

In the region $r < 1.5D_0$, however, the drawdown in observation wells that are screened throughout the aquifer is always greater than that of the water table, the difference being greater at smaller values of r. Equation (94) describes (Section V, C, 2) the average drawdown in observation wells induced by the flow toward steady wells in sloping leaky water-table aquifers. A special case of this equation is that for a horizontal ($1/\beta = 0$) nonleaky ($1/B = 0$) water-table aquifer. Thus, from Eq. (94), with $1/\beta = 0$ and $1/B = 0$, the average drawdown in observation wells in a horizontal nonleaky aquifer, if $D_0 - h_w < 0.5\, D_0$ and $t > 30r_w^2/\nu$, is given by

$$D_0^2 - \bar{D}^2 = (Q/2\pi K)W(u) \qquad (92)$$

For relatively large time $u < 0.05$ [see Section II, C for approximation of $W(u)$], or $t > 5r^2/\nu$, Eq. (92) may be written as

$$D_0^2 - \bar{D}^2 = (Q/\pi K) \ln(1.5\sqrt{\nu t}/r)$$

in which $u = r^2/4\nu t$, $\nu = K\bar{b}/\epsilon$, and $\bar{b}$ is a weighted average of the depth of the flow, which may, for all practical purposes, be taken as equal to D_0 in the region $r > 1.5D_0$, where s/D_0 is generally small. In regions very close to the pumping well, where the depth of saturation as computed by Eq. (92) approximates (Section *V*, A, 1, *b*) more closely the depth of water in piezometers open at the base of the aquifer (smaller than the actual average depth in observation wells), the value of $\bar{b}$ may be taken as equal to D_w (the depth of saturation at the face of the well), since this approximation hypothetically increases the specific yield and, consequently, increases the calculated value of $\bar{D}$, thus, making it closer to the true value of the average depth in the observation well. An estimate of the value of D_w may be obtained by using Eq. (89) (see Example 19) or from other procedures [36].

In the immediate vicinity of the well, $\bar{D}$, as calculated from Eq. (92), deviates appreciably from its actual value in observation wells. In this region it may be better approximated by $\bar{D} = 0.5(D_t + D_b^2/D_t)$ in which D_t (the height of the water table) and D_b (the height of water in a piezometer open at the base of the aquifer) are the values of D and $\bar{D}$ as given by Eqs. (90) and (92), respectively.

For $\tau > 5$, when the effects of the vertical component of the velocity on the main horizontal flow become negligible, and for $r > 1.5D_0$, where the effect of the curvilinearity of the flow is very small, Eq. (90) (see approximation of *V*) and Eq. (92) become the same, as should be expected.

The left-hand member of Eq. (92) may, in terms of the drawdown, be written as $2D_0 s(1 - s/2D_0)$, which, if $s/D_0 < 0.02$, may be approximated by $2D_0 s$. Consequently, the Theis equation results.

If observational data of drawdown in wells that are screened throughout the aquifer are available, the height of the water table D in the region $r < 1.5D$ may be calculated from Eq. (92) after replacing $D_0^2 - \bar{D}^2$ with $2D\bar{D} - D^2$; $\bar{D}$, being the depth of water in the observation wells, is obtained from field measurements.

When $t < 30r_w^2/\nu$, the drawdown equation, by comparing Eq. (92) with Eq. (63) and, consequently, with Eq. (64), may be deduced as

$$D_0^2 - \bar{D}^2 = (Q/2\pi K)S(\nu t/r_w^2, r/r_w).$$

3. *Partially Penetrating Wells*

When the drawdown is small relative to the saturated thickness D_0, the drawdown around partially penetrating wells may be approximated by the equations derived for artesian conditions (Section IV) after replacing s, b, and u and B therein with $s - s^2/2D_0$, D_0, and u and B (in terms of water-table aquifer parameters), respectively. The top of the artesian aquifer is replaced by the static position of the water table. For the case of deep aquifers or for relatively short times, s of *Eqs.* (82) and (84) should be replaced by $s-s^2/2l$, other replacement being the same. This approximation is possible since the boundary-value problem describing the flow in water-table aquifers (for small s/D_0) is analogous to that for artesian aquifers.

C. Flow to Wells in Sloping Leaky and Nonleaky Aquifers

1. *Equation of Motion*

Figure 15 illustrates ground-water flow in a sloping water-table aquifer resting on a semipervious layer through which vertical leakage in proportion to the drawdown of the water table takes place. This layer overlies an artesian aquifer in which the water levels are not influenced by pumping in the water-table aquifer. The storage in the semipervious layer is neglected. This leaky flow system is a special case of that governed by Eq. (16); namely, that of Fig. 3. In the present system $f = ix$, $w = 0$, $H = f + D(x,y,t)$, $b = H - f = D$, $\bar{F} = 0$ and the rate of vertical leakage $u = (K'/b')(D_a - D)$; where D and D_a are, respectively, the heights above the base of the aquifer of the water table and the piezometric surface (assumedly uninfluenced by pumping in the water-table aquifer) of the aquifer supplying leakage. Consequently, $D = H-f$, $\bar{\varphi} = \bar{D}(x,y,t) + f + c$, $\varphi(f) = D_b(x,y,t) + f + c$, and, from boundary conditions on a water table, $\varphi(H) = H + c = D + f + c$; where $\bar{\varphi}[=(1/D)\int_f^H \varphi(x,y,z,t)\,dz]$ is the average head along the depth of flow, $\bar{D}$ is the depth of water in an observation well that is perforated throughout the aquifer, and D_b is the depth of water in a piezometer that is open at the base of the aquifer.

Substitution of these relations in Eq. (16) and making use of the assumptions that $S_s <<< \epsilon$ and that $D_b \simeq D \simeq \bar{D}$ when $D_0 - h_w < 0.5D_0$ and $i < 0.02$ will yield, after some manipulation, the following differential equation

$$\partial^2\bar{D}^2/\partial x^2 + \partial^2\bar{D}^2/\partial y^2 + 2i\,\partial\bar{D}/\partial x + (2K'/b'K)(D_a - \bar{D}) = (2\epsilon/K)\partial\bar{D}/\partial t$$

Let $D_0(x,y,\tau_0 + t)$ be the distribution of the depth of flow that would prevail in this flow system if the drainage facility (such as a drain or a well) were not discharged; D_0 is assumedly a solution of the preceding differential equation. Also, let the drainage facility begin to discharge at the instant τ_0 and let the depth of water thereafter, in completely penetrating idle wells, be represented by $\bar{D}(x,y,t)$. Subtracting the preceding differential equation from

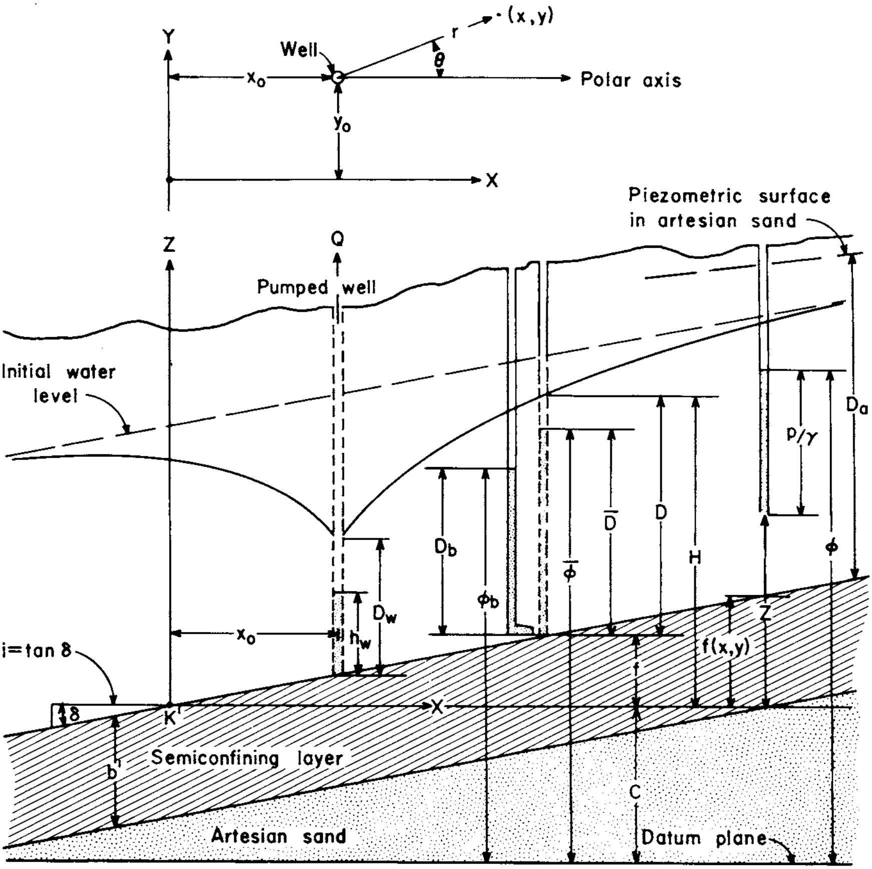

FIG. 15. Diagrammatic representation for the flow to a well in a sloping leaky water-table aquifer.

itself after replacing $\bar{D}$ with D_0 and linearizing the resulting equation by arbitrarily making the quantity $0.5(D_0 + \bar{D})$ equal to a constant $\bar{b}$ will give the following equation

$$\partial^2 Z/\partial x^2 + \partial^2 Z/\partial y^2 + (2/\beta)\partial Z/\partial x - (1/B^2)Z = (1/\nu)\partial Z/\partial t \tag{93}$$

in which

$$Z = D_0^2 - \bar{D}^2, \qquad \beta \simeq 2\bar{b}/i, \qquad \nu \simeq K\bar{b}/\epsilon, \qquad B^2 \simeq K\bar{b}/(K'/b')$$

and

$$\bar{b} = 0.5(D_{0m} + \bar{D}_m)$$

where $\bar{D}_m$ is a weighted mean of the areal distributions of the depth of saturation during the period of induced flow and D_{0m} is a weighted mean of the areal distributions during the same period of the depth of saturation if the

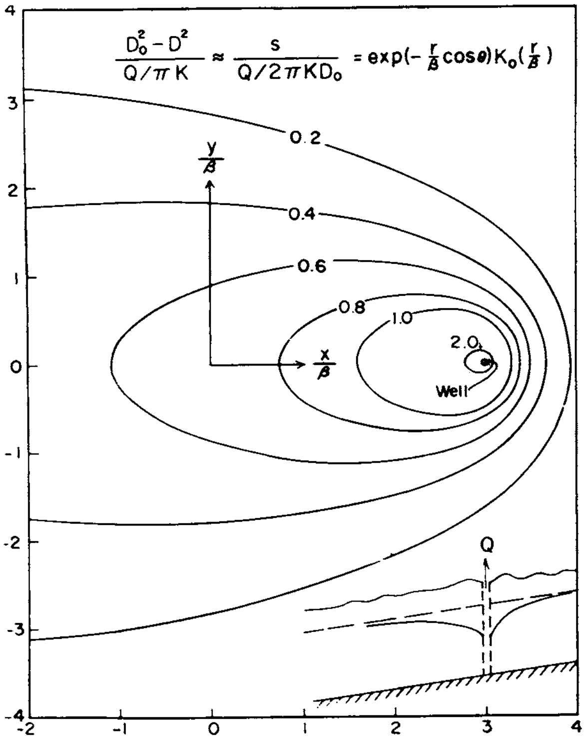

FIG. 16. Drawdown distribution around a steady well in a sloping nonleaky water-table aquifer.

drainage facility were not discharged. The constant $\bar{b}$ may be approximated by

$$\bar{b} \simeq 0.5[D_{0w}(\tau_0) + D_{0w}(\tau_0 + t_0)], \qquad \text{at points} > 1.5D_0$$

and by

$$\bar{b} \simeq 0.5[D_w(0) + D_w(t_0)], \qquad \text{at points} < 1.5D_0$$

where t_0 is the period (or subperiod if it becomes necessary to use the method of successive approximation) at the end of which $\bar{D}$ is to be estimated.

Equation (39) closely describes the effect of induced flow in sloping leaky water-table aquifers, provided $D_0 - h_w < 0.5D_0$ and $i < 0.02$, h_w being the depth of water in (not at the face) of the completely penetrating drainage facility. For points located at distances $> 1.5D_0$ from the drainage facility, $\bar{D}$ represents the height of the water table above the base of the aquifer, and for points at distances $< 1.5D_0$, $\bar{D}$ represents the depth of water in idle wells screened throughout the depth of saturation (Section V, A, 1, *b*).

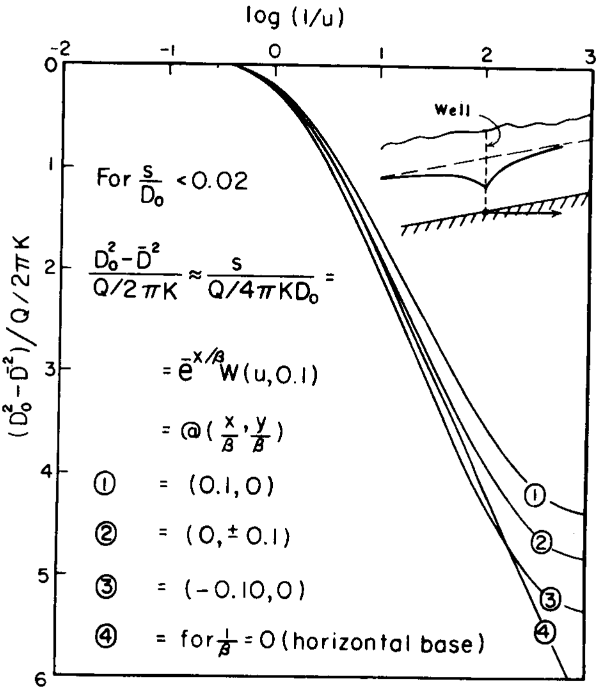

FIG. 17. Time-drawdown variation around a steady well in a sloping nonleaky water-table aquifer.

2. *Drawdown Equation*

The drawdown distribution around steady wells in sloping leaky water-table aquifers is described approximately by Eq. (93) and the following conditions: $D(r,\theta,0) = D_0(r,\theta,\tau_0)$, $D(\infty,\theta,t) = D_0(\infty,\theta,\tau_0 + t)$, and $r\,\partial Z/\partial r = -Q/\pi K$ as $r \to 0$, where r and θ are the polar coordinates with the pole at the center of the well. The last condition is obtained from $Q = K\int_0^{2\pi} r_w d\theta \int_f^h [\partial\varphi(r_w,\theta,z,t)/\partial r]dz$ by making use of the same assumptions and approximations used in obtaining Eq. (93), recalling that $\bar{D}$ at the face of the well is more or less independent of θ and that $\int_0^{2\pi} r_w(\partial D_0^2/\partial r)d\theta = 0$.

By making the substitution $Z = Z'(x,y,t)\exp[-(x-x_0)/\beta]$ the boundary-value problem, in terms of Z', becomes analogous to that of Example 10; the well is located at the point (x_0,y_0). Consequently, the required solution (valid for $i < 0.02$; $D_0 - h_w < 0.5\,D_0$, and $t > 30r_w^2/\nu[1-(10\alpha r_w)^2]$, with $\alpha r_w < 0{\cdot}1$) may be obtained as

$$D_0^2 - \bar{D}^2 = (Q/2\pi K)\exp\left(-\frac{r}{\beta}\cos\theta\right)W(u,\alpha r) \tag{94}$$

in which

$$u = r^2/4\nu t \qquad \text{and} \qquad \alpha^2 = 1/\beta^2 + 1/B^2$$

The corresponding *steady-state solution*, from Eq. (94), with $t = \infty$ or $u = 0$, is readily obtained as

$$D_0^2 - \bar{D}^2 = (Q/\pi K) \exp\left(-\frac{r}{\beta}\cos\theta\right) K_0(\alpha r)$$

For definitions, tables, and useful approximations of the functions in the above equations, refer to Section II, C.

When evaluated at $r = r_w$, Eq. (94) and the equations derived therefrom will give the drawdown (neglecting well losses) in the well (not at the face of the well). They will closely represent the drawdown of the water table at the well face if $h_w/D_0 >> 0.5$.

The equations of drawdown around wells in sloping nonleaky, horizontal leaky, and horizontal nonleaky aquifers may be readily obtained from the above equations by letting $1/B = 0$, $1/\beta = 0$, and $\alpha = 0$, respectively.

The effects of a sloping base are shown in Figs. 16 and 17. In regions in which $r/\beta > 0.01$, the distribution of drawdown does not have the usual circular pattern that occurs around wells in horizontal aquifers; it has a pattern similar to that shown in Fig. 16. Within the period $t < 2.5r\beta/\nu$, the drawdown variation with the logarithm of time (Fig. 17) at points in the region $r/\beta > 0.01$ has the same general trend of variation (not necessarily equal to) as that which obtains if the aquifer is horizontal. This variation will be the same in the two aquifers only for points that lie on a line passing through the center of the well and parallel to the y-axis. For $t > 2.5r\beta/\nu$, this variation at any point of the flow system of Fig. 17 resembles that which would have occurred if the aquifer were of a horizontal base and were connected to some source of recharge, such as beds of streams or lakes (Fig. 19) or leakage from or through semipervious layers (Fig. 8). It also resembles the effects of partial penetration of wells in horizontal aquifers (Figs. 11 and 12).

D. Recovery Equations

By a procedure similar to that used in solving Example 10, the water levels around a gravity well during recovery may be obtained as

$$\left\{\begin{array}{l}\text{The left-hand member of any of}\\ \text{the unsteady drawdown equations}\end{array}\right\} = Z(t) - Z(t') \tag{95}$$

in which $Z(t)$ is the right-hand member of the unsteady drawdown equation of the particular flow system and $Z(t')$ is the value of $Z(t)$ after replacing t with t', t' being the time since cessation of pumping, or $t' = t - t_0$, where t_0 is the period of continuous pumping, and t is the time since pumping began. Thus, the recovery equation for a well in a sloping leaky aquifer will, from Eq. (94), be given by

$$D_0^2 - \bar{D}^2 = (Q/2\pi K)\exp\left(-\frac{r}{\beta}\cos\theta\right)[W(u,\alpha r) - W(u',\alpha r)]$$

and that for the recovery of the water table in a horizontal nonleaky aquifer will, from Eq. (90), be given by

$$s = (Q/2\pi T_0)(1 + C_f)V(\tau,\rho) - (1 + C_f')V(\tau',\rho)$$

where u',τ' are the values of u and τ after replacing t with t' and C_f' (since C_f depends on τ) is the correction factor during recovery.

Example 19. The average depth of saturation in a horizontal nonleaky aquifer is 100 ft. From aquifer tests on wells located at $r > 1.5D_0$ ($>150\,ft$) from a pumped well, the formation coefficients are estimated as $T_0 = KD_0 = 0.1$ ft²/sec and $\epsilon = 0.1$. A 24-in. well is pumped at the rate of 3.14 ft³/sec for about 3 days (25×10^4sec), then the well is shutdown. Determine the following just before shutdown:

(1) The drawdown in the well assuming zero well losses;

(2) the approximate height of seepage face of the pumping well;

(3) the drawdown of the water table at the site of an observation well located at $r = 10$ ft;

(4) the drawdown in a piezometer open at the base of the aquifer at the site of the observation well;

(5) the drawdown in the observation well using $\bar{D} = 0.5(D_t + D_b^2/Dt)$, and compare it with that obtained by using Eq. (92), with $\bar{b} = D_w$;

(6) the extent of the zone of influence of the well; and

(7) the time required for the water level in the well to recover after shutdown, to about 0.1 ft from its original level.

(1) Since $\tau = Kt/\epsilon D_0 = 25 > 5$, from Eq. (91) (with $\rho_w = 0.01$), $h_w = 58.2$ ft, and $s_w = 41.8$ ft.

(2) Equation (89) may be used to estimate an approximate value of D_w, provided t is long enough so that $t > 5\,\epsilon\, r_i^2/KD_0$, or $\tau > 5(r_i/D_0)^2$, and provided that $r_i > 1.5D_0$. This is because the shape of the water table at this time [see special case of Eq. (92)] varies with the logarithm of r from the face of the well to at least $r = r_i$, and hence the instantaneous flow system may be approximated by the hypothetical system of Fig. (13), on which Eq. (89) is based. At the end of the period of pumping, $\tau = 25$. Since r_i/D_0 is to be $\geq 1.5D_0$, and $\tau > 5(r_i/D_0)^2$, then r_i may be selected, in the range $1.5 \leq r_i/D_0 \leq \sqrt{5}$, for use in Eq. (89). For $r_i/D_0 = \rho_i = 2$ and $\tau = 25$, Eq. (90) with $C_f \approx 0.03$, gives $D_0 - D_i = 6.9$ ft or $D_i = 93.1$ ft. For $r_i = 200$ ft, $D_i = 93.1$ ft, $h_w = 58.2$ ft, and $r_w = 1$, Eq. (89) gives $D_w - h_w \approx 20.2$ ft and $D_w = 78.4$ ft.

(3) For $\tau = 25$, $\rho = 0.1$, and $C_f \approx -0.137$, Eq. (90) gives, as the drawdown of the water table, $s = 18.6$ ft, and $D_t = D_0 - s = 81.4$ ft.

(4) Although it is used to approximate the average drawdown in observation wells at $r < 1.5D$, Eq. (92) approximates more closely the head distribution at the base of the aquifer. Thus, with $u = \rho^2/4\tau = (0.1)^2/4(25) = 10^{-4}$, Eq. (92) with D_b replacing $\bar{D}$, gives $D_b \approx 75.4$ ft, or $s_b = 24.6$ ft.

(5) $D = 0.5(D_t + D_b^2/D_t) = 75.7$ ft, or $\bar{s} = 24.3$ ft. By using Eq. (92) with $\bar{b} = D_w$ (that is for $u = r^2\epsilon/4KD_wt = (r^2/D_0D_w)/4\tau = 1.27 \times 10^{-4}$) the result is $\bar{D} = 76$ ft, or $\bar{s} = 24$ ft.

(6) The zone of influence extends to a value of r that makes the drawdown computed by Eq. (90) or (92) effectively zero, say $s = 0.01$ ft. Since $\bar{D} \approx D_0$, then from Eq. (92), the drawdown $D_0 - \bar{D} = [Q/4\pi KD_0]W(u) = 0.01$, or $W(u) = 0.004$. From tables of $W(u)$, $u = 4$, or $r^2\epsilon/4KD_0t = 4$, from which the radius of influence $r \approx 2000$ ft.

(7) From Eqs. (95) and (91), the equation for water-level recovery in the well is $D_0^2 - h_w^2 = Q/\pi K[\ln(1.5\sqrt{\tau}/\rho_w) - \ln(1.5\sqrt{\tau'}/\rho_w)]$ or $s_w - s_w^2/2D_0 = (Q/2\pi KD_0)\ln(t/t')$. Thus, for $s_w = 0.1$ ft, $\ln(t/t') \approx 0.1$, or $t/t' = \exp(0.1) \approx 1.10$. Hence $(t_0 + t')/t' = 1.10$ or $t' = 10t_0 = 10 \times 25 \times 10^4$ sec ≈ 29 days.

VI. Interfering Wells

Ground water is often extracted by more than one well. Unless the periods of continuous operation are relatively short and/or the spacing of the wells in a well field is so great that their zones of influence do not effectively overlap, the discharge or the drawdown of individual wells is affected by the neighboring wells.

A. Well Fields

A group of wells operating in a given area constitute a well field.

1. Drawdown Around Interfering Wells

Because the differential equation of motion is linear in the dependent variable (the head φ, the drawdown s, or the square of depth of the flow D^2), a linear combination of its solutions is also a solution. Consequently, since in a well field consisting of N steadily discharging artesian wells, each of the well potentials separately satisfies the well boundary-value problem, and since the boundary condition at the face of each well is not influenced (singularities do not contribute to the value of a line integral if they are not enclosed by the path of integration) by the existence of the other wells, the distribution of drawdown is given by

$$s = \sum_{i=1}^{N} Q_i Z_i(r_i, t_i) \tag{96}$$

and the distribution of water levels around corresponding water-table wells is obtained from

$$D_0^2 - \bar{D}^2 = \sum_{i=1}^{N} Q_i Z_{wi}(r_i, \theta_i, t_i) \tag{97}$$

in which, $Q_i Z_i$, henceforth called the *artesian well potential*, is the right-hand member of the drawdown equation of a particular flow system (Sections III and IV; also Boulton's solution in Section V), $Q_i\, Z_{wi}$, *the water-table well potential*, is the right-hand member of the equations giving the depth of flow (in terms of $D_0^2 - \bar{D}^2$) in a water-table aquifer (Section V), Q_i is the discharge (may be positive, or negative) of the *i*th well, r_i is the distance from the *i*th well to the point of observation, θ_i is the polar angle with the pole at the center of the *i*th well, and t_i is the time measured from the instant the *i*th well is put into operation.

2. *Discharge of Interfering Wells*

If the location of each of N wells is known and the water levels in each of the N wells at the end of a given period of continuous pumping is preassigned, the discharge of each well can be obtained by solving the N linearly independent equations written for the water level in each of the wells, using Eq. (96) or (97) as the case may be. Thus, two artesian wells a distance m apart, discharging simultaneously over the same period of time t_0 from a nonleaky aquifer, and having the same diameter $2r_w$ and drawdown s_w will have discharges Q_1 and Q_2, from Eqs. (63) and (96), given by

$$Q_1 = Q_2 = 4\pi T s_w / [W(r_w^2/4\nu t_0) + W(m^2/4\nu t_0)]$$

Similarly, for three wells forming an equilateral triangle a distance m on a side,

$$Q_1 = Q_2 = Q_3 = 4\pi T s_w / [W(r_w^2/4\nu t_0) + 2W(m^2/4\nu t_0)]$$

If t_0 is long enough that $m^2/4\nu t_0 < 0.05$, the expressions for the preceding particular well patterns may be given, respectively, by

$$Q_1 = Q_2 = 2\pi T s_w / \ln(2.25\nu t_0 / m r_w)$$

and

$$Q_1 = Q_2 = Q_3 = 2\pi T s_w / \ln(R^3 / r_w m^2)$$

where $R = 1.5\sqrt{\nu t_0}$.

The discharge of each of four wells forming a square of side m, provided $m^2/2\nu t_0 < 0.05$, is given by

$$Q_1 = Q_2 = Q_3 = Q_4 = 2\pi T s_w / \ln(R^4 / r_w m^3 \sqrt{2})$$

and for a line of three equally spaced wells a distance m apart, provided

$m^2/\nu t < 0.05$, the discharge of each of the outer wells is

$$Q_1 = Q_3 = [2\pi T s_w \ln(m/r_w)]/f(R, m, r_w)$$

and the discharge of the middle well is

$$Q_2 = [2\pi T s_w \ln(m/2r_w)]/f(R, m, r_w)$$

where $f(R, m, r_w) = 2 \ln(R/m) \ln(m/r_w) + \ln(m/2r_w) \ln(R/r_w)$.

The corresponding equations for a nonleaky, horizontal, water-table aquifer are obtained from the preceding expressions by merely replacing $2\pi T s_w$ with $\pi K(D_0^2 - h_w^2)$.

3. *Effect of Well-Field Operation over an Area*

When estimates of the drawdown near a pumped well are being made, it is necessary to include the influence of other wells operating in the area. If the locations and the discharges of each of the wells in the area are known, the total lowering of the water levels at a given location can be found by using Eq. (96) or (97), whichever applies. In addition to being laborious, such a procedure requires more data than are ordinarily available. An estimate of the effect of a large number of wells on the water levels within and outside the well field may be obtained by idealizing the problem by assuming that the pumping of the well field is uniformly distributed over a circular area of radius a within which most of the wells are located. Such a situation may be effectively realized if pumping for irrigation and drainage is accomplished by use of a large number of wells distributed throughout the area. Thus, if the total discharge of the well field is V in units of volume per unit time, the water level at any given point owing to a continuous distribution of artesian wells over the whole circular area may be obtained from

$$4\pi T s = \int\int (V/\pi a^2) f(r, r_1, \theta, \theta_1, t) r_1 \, dr_1 \, d\theta_1$$

where (r,θ) is the location in polar coordinates of the point at which water level is desired and (r_1,θ_1), the location of a well of discharge $(V/\pi a^2) r_1 \, dr_1 d\theta_1$, the pole being at the center of the well field and the integration being over the whole circular area. The function f is the well function for a given flow system.

For *leaky aquifers* without storage in semipervious layers, the drawdown expressions during steady-state flow are as follows:

for $r < a$:

$$4\pi T s = 4V(B^2/a^2)[1 - (a/B) I_0(r/B) K_1(a/B)]$$

for $r > a$:

$$4\pi T s = 4V(B/a) I_1(a/B) K_0(r/B)$$

For nonleaky aquifers, the corresponding expressions are

for $r < a$ and $t > 0.4r^2/\nu$:

$$4\pi Ts = V\{W(u_a) + (1/u_a)[1 - \exp(-u_a)] - (r/a)^2 \exp(-u_a)\} \qquad (98)$$

for $r > a$ *and* $t > 0.4a^2/\nu$:

$$4\pi Ts = V\{W(u) + (0.5u_a)\exp(-u)\}$$

in which $u_a = a^2/4\nu t$ and $u = r^2/4\nu t$.

If the well field taps a horizontal water-table aquifer, the corresponding expressions are obtained by replacing $4\pi Ts$ with $2\pi K(D_0^2 - \bar{D}^2)$.

B. Spacing of Wells

1. *Economical Spacing of Wells*

The proper spacing of wells involves economics as well as hydrologic consideration. The farther apart the wells are, the less their interference, but the greater the cost of connecting pipelines and power installation. The cost of a plant for a group of wells, insofar as it is affected by the spacing of the wells, may be reduced to an annual unit charge consisting of (1) the yearly cost of lifting the water against the additional head caused by well interference; and (2) the cost of connecting pipelines between the wells and the power installation. The latter includes maintenance and depreciation, as well as taxes and interest on borrowed capital, all of which may be expressed as an annual cost per unit length of intervening distances. The total drawdown in one pumped well caused by pumping other nearby wells can be obtained by using Eq. (96) or (97) (whichever is applicable to the flow system under consideration). Consequently, the total additional head caused by well interference can be expressed in terms of the unknown intervening distances, the assumedly known hydraulic properties, and the parameters defining the geometry of the flow system.

A detailed treatment of this subject and a list of expressions for the "optimum spacing parameter" of various grouping of wells in different flow systems are available in the literature [37].

Optimum well spacing: The general expression for optimum well spacing is obtained as follows:

Excluding the constant cost of lifting the water against the head developed in each of the wells when it is operating alone, the yearly cost of operating a well field may, by considering average cost values over a year's time, be expressed as

$$C = c'm\delta + c'' \sum_{n=1}^{N} Q_n \int_0^{t_0} s_n \, dt$$

in which C = total yearly cost of operation as affected by well interference;

c' is the capitalized cost per unit length of pipe line for maintenance, depreciation, original cost of pipe line, etc.; c'' is the cost to raise a unit volume of water a unit height, consisting largely of power charges, but also properly including some additional charges on the equipment; s_n is the total drawdown in the nth well caused by pumping all the other wells; $m\delta$ is the length of connecting pipelines between wells and the power installation; N is the number of wells in operation; Q_n the discharge of the nth well, in unit volume per unit time; t_0 is the period of continuous pumping; m is the "spacing parameter" which is the distance between any two wells—other intervening distances being expressed in terms of m; and δ is a constant when multiplied by m gives the length of connecting pipelines.

The minimum cost will correspond to that point at which the first derivative of C with respect to m equals zero. By differentiating the above equation and equating the resulting expression to zero, the following relation obtains

$$\sum_{n=1}^{N} Q_n \int_0^{t_0} (\partial s_n/\partial m)\, dt = -(c'\delta/c'') \tag{99}$$

The optimum spacing parameter m_0 is the value of m that satisfies the above equation. If the period of continuous pumping t_0 is more than one year, the right-hand member of the preceding equation should be multiplied by t_0/t_0' where t_0' is a period of one year.

If, within the year, there occurs M periods of well operation, each followed by a period of shutdown long enough for the water levels to effectively recover to their original level, the integral with respect to t in the left-hand member of Eq. (99) is replaced by M integrals the upper limit of each being equal to t_M, the duration of the Mth period of continuous pumping.

The use of Eq. (99) will be illustrated for a group of wells in nonleaky aquifers. For its use in other systems, reference is made to [37].

For a nonleaky artesian aquifer, the drawdown equation is given by $s = (Q/4\pi T)W(u)$. From this equation, the following expression is obtained

$$s' = \int_0^{t_0} (\partial s/\partial m)\, dt = c\int_0^{t_0} (\partial s/\partial r)\, dt = -(Qt_0/2\pi Tm) \cdot [\exp(-u_0) - u_0 W(u_0)]$$

in which $u_0 = r^2/4\nu t_0$, $r = cm$ where c is a constant relating the spacing parameter m to r. After a relatively long period of continuous pumping, the value of u_0 (r equaling the longest distance between any two of the N wells) is generally less than 0.05, in which case, the above relation can very closely be approximated by $s' = -Qt_0/2\pi Tm$. Thus, if there are N wells in the field, the value of s_n' will be given by

$$s_n' = -(t_0/2\pi Tm)\left\{\sum_{i=1}^{N} Q_i, \quad \text{with} \quad Q_i = 0 \quad \text{for} \quad i = n\right\}$$

When this is put in Eq. (99), and when the resulting expression is solved for m, the optimum spacing parameter m_0 will be

$$m_0 = (c''t_0/2\pi c'T\delta)\sum_{n=1}^{N} Q_n\left\{\sum_{i=1}^{N} Q_i, \quad \text{with} \quad Q_i = 0 \quad \text{for} \quad i = n\right\}$$

If there are M periods of pumping within the year [see paragraph following Eq. (99)] the shortest of which is such that $u_m < 0.05$, the value of t_0 in the preceding expression for m_0 will be given by $t_0 = t_1 + t_2 + \cdots + t_M$, where u_m is the value of u_0 with t_M replacing t_0.

Consider a group of three wells. Let the distance between the first and second wells be m, that between the first and third $\alpha' m$, and that between the second and third $\alpha'' m$, where α' and α'' are constants converting the actual distances in terms of m. Also, let Q_1, Q_2, Q_3 be the discharges of the three wells, respectively, and $m\delta$ the length of the connecting pipelines (usually the shortest path connecting the wells; in the present case along the medians of the triangle formed by the wells), δ being a constant dependent on α' and α''. The number of wells N, being equal to three, the preceding equation gives

$$m_0 = (c''t_0/\pi c'\ \delta T)(Q_1Q_2 + Q_1Q_3 + Q_2Q_3) \qquad (100)$$

which for three wells having the same discharge and forming an equilateral triangle a distance m on a side (here $\alpha' = \alpha'' = 1$, and if the connecting pipelines are taken along the medians of the triangle, $\delta = \sqrt{3}$), becomes

$$m_0 = c''\sqrt{3}t_0Q^2/c'\pi T$$

For a line of three equally spaced wells, a distance m apart, and each having the same discharge (here $\alpha' = 2$, $\alpha'' = 1$, and $\delta = 2$), Eq. (100) becomes

$$m_0 = 3c''t_0Q^2/2c'\pi T.$$

For two interfering wells, m_0 can be computed from the general expression, with $Q_3 = 0$ and $\delta = 1$.

2. *Spacing of Wells for Permissible Drawdown*

The maximum allowable drawdown in a well is often determined by field and constructional conditions rather than by economic considerations. Physical limitations of the aquifer (such as depth, thickness, or piezometric head), limitation on available pumping equipments (such as suction lift, setting of bowls, or horsepower of motors), and depths of existing wells are among factors that may limit the total permissible drawdown in a well.

Given N wells forming a definite geometrical pattern, the distances between the wells can be expressed in terms of a spacing parameter that is the distance between any two wells in the group. Given the discharge and schedule of operation of each of the wells during peak demand, the well in which the maximum drawdown develops can be located by using Eq. (96) or (97) (whichever applies) and an assumed spacing parameter, since the point at which maximum drawdown occurs is independent of the magnitude of the spacing parameter. Having determined the maximum allowable drawdown, the well in which this drawdown will occur, and the schedule of operation of the well field, Eq. (96) or (97) (whichever applies), written for the drawdown (increased by the well losses, if necessary) at this well, will afford the means for solving for the "spacing parameter," which is the only unknown in the resulting equation. The solution of this equation (the equation of interference) may have to be obtained by trial, using the appropriate table of the function involved therein. This procedure is not restricted to wells in an infinite aquifer; it is applicable also to wells near other boundaries (Section VII), the drawdown distribution around which is a problem of interfering discharging ($+Q$) and hypothetically recharging ($-Q$) and/or discharging wells.

In certain situations, the solution of the equation of interference is direct. For example, for a group of N wells pumping simultaneously from an artesian aquifer and forming a circular battery of $N-1$ wells, with the maximum drawdown occurring in the Nth well which is located at the center of the circular battery, the spacing parameter m, which, for this grouping, is conveniently taken as the radius of the circular battery, is, from Eq. (96) and Eq (63), given by

$$W(u_m) = [4\pi T s_N - Q_N W(u_w)] \Big/ \left[\sum_{i=1}^{N-1} Q_i\right] \tag{101}$$

in which $u_w = r_w^2/4\nu t_0$, $u_m = m^2/4\nu t_0$, and s_N is the permissible drawdown in the Nth well at the end of a continuous period t_0 of simultaneous pumping. From tables of the function W, the value of u_m corresponding to the known value of W, as computed from the right-hand member of the preceding expression, can be obtained. Consequently, the value of m is obtained from this value of u_m.

If this group of wells taps a leaky artesian aquifer, and if the period of pumping is sufficiently long to assure steady-state flow, the expression corresponding to the preceding equation will be

$$K_0(m/B) = [2\pi T s_N - Q_N K_0(r_w/B)] \Big/ \left(\sum_{i=1}^{N-1} Q_i\right)$$

The solution for m can be readily performed, using a table of the function K_0.

Another example is that of a group of N wells pumping simultaneously from a nonleaky aquifer for which the period of continuous pumping is such that $u = R^2/4\nu t_0 < 0.05$, where R is the distance between the well farthest from the well in which the maximum drawdown occurs. For this situation, [since for

$u < 0.05$, $W(u) \approx \ln(0.562/u)$] the spacing parameter may be computed from Eq. (96) as

$$\ln m = \left(1/\sum_{i=1}^{N-1} Q_i\right)\left\{\left[-Q_N \ln r_w + \ln(1.5\sqrt{\nu t_0})\sum_{i=1}^{N} Q_i - 0.5\sum_{i=1}^{N-1} Q_i \ln d_i\right] - 2\pi T s_N\right\} \tag{102}$$

in which $d_i = r_i/m$, a constant known from the geometry of well locations, that relates r_i (the distance between the ith well and the well (Nth well) at which the maximum drawdown occurs) to the spacing parameter m; other symbols have been defined.

After calculating m, verification of the condition $R^2/4\nu t_0 < 0.05$ should be made.

Example 20. A well field consisting of about two hundred wells uniformly distributed within an area that may be approximated by a circle of radius 5 mi, or approximately 30,000 ft. The well field taps a nonleaky artesian aquifer, effectively infinite in areal extent, having a value of $T = 0.067$ ft²/sec and a value of $S = 10^{-3}$. If the wells are to be pumped continuously for a period of 50 days at a total rate of about 60 ft³/sec; (1) estimate the drawdown at the end of operation in a well, located at 25,000 ft from the center of the well field, in addition to that caused by its own pumping; (2) estimate the same for a well located at center of the field.

(1) For $r = 25{,}000$, and $\nu = T/S = 67$, the value of $t (= 50 \text{ days} = 4.32 \times 10^6 \text{ sec}) > 0.4r^2/\nu = 3.73 \times 10^6$ sec. Thus, Eq. (98) is applicable which for $u_a = a^2/4\nu t = (9 \times 10^8)/(4)(67)(4.32 \times 10^6) = 0.78$, $r/a = 0.833$, $V = 60$ ft³/sec, and $4\pi T = 0.84$, and with the use of appropriate tables gives $s \approx 50$ ft.

(2) The corresponding value (for $r = 0$) is $s \approx 73$ ft.

Example 21. A pumping plant is to be composed of three wells forming an equilateral triangle. The plant is to tap a nonleaky artesian aquifer of $T = 0.067$ ft²/sec and $S = 10^{-4}$. If the schedule of operation for the whole year is 2 weeks of continuous pumping followed by a shutdown of 3 weeks, and if the discharges of the wells are, respectively, 20, 50, and 100 ft³/min, (1) estimate the most economical spacing, and (2) the drawdown at the end of 2 weeks in the well whose discharge is 100 ft³/min, if $r_w = 1$ ft.

(1) Let the cost of lifting water c'' be 10^{-6} dollars per cubic foot of water per foot of lift. Let the cost of pipeline and electric wiring be 10 dollars per foot, and capitalize it at 10% per year. The value c' will be 1 dollar per foot for capital charges, depreciation, and maintenance during the year. Let the connecting pipeline be along the medians of the equilateral triangle, or $\delta = \sqrt{3}$. Thus, for $t_0 (= t_1 + t_2 + \cdots) = 10 \times 14 \text{ days} \approx 2 \times 10^5$ min, $T = 4$ ft²/min, and for the values of c'', c', δ, Q_1, Q_2, and Q_3, as given above, the

optimum spacing parameter is computed from Eq. (100) as $m_0 \approx 74$ ft. Equation (100) is valid if $u = m_0^2/4\nu t_M < 0.05$, ($t_M = 14$ days $= 2 \times 10^4$ min), hence it is valid for an aquifer whose storage coefficient $S < 0.2Tt_M/m_0^2$, or for $S < 2.92$. Consequently the validity of the above computation is assured since $S = 10^{-4}$ for the aquifer tapped by the wells.

(2) From Eq. (96), the required drawdown (excluding well loss) is given by $s = (1/4\pi T)\,[100W(u_w) + (20 + 50)W(u_m)]$. For $u_w = r_w^2/4\nu t_M \approx 3.1 \times 10^{-10}$ and $u_m = m_0^2/4\nu t_M \approx 1.7 \times 10^{-6}$, this equation gives $s \approx 60.2$ ft.

Example 22. If the permissible drawdown in the wells of the preceding example is not to exceed 55 ft at the end of 2 weeks of continuous pumping, determine the allowable spacing of the wells.

The maximum drawdown will occur in the 100 ft^3/min well. From Eq. (101), for $N = 3$, $u_w = 3.1 \times 10^{-10}$, $W(u_m) = [4\pi(4)(55) - 100W(u_w)]/(20 + 50) \approx 9.0$. From tables of W, $u_m \approx 6.9 \times 10^{-5}$; consequently, $m^2 = r_w^2(u_m/u_w) = (1)(6.9 \times 10^{-5}/3.1 \times 10^{-10})$, or $m \approx 470$ ft. Equation (102) may be used also, with $N = 3$ and $d_1 = d_2 = 1$.

VII. Wells near Other Boundaries

Frequently, wells are located near streams, canals, lakes, seas, and other bodies of surface and ground water that are hydraulically connected to the aquifer they are pumping from. Other wells are located near tight faults, buried rock valleys, or dikes and similar structures that cut the aquifer, thus preventing ground-water flow across such barriers. Unless the periods of continuous pumping of all such wells are relatively short, their zones of influence will eventually extend to these boundaries, at which time the formulas for flow around wells in an infinite aquifer become inapplicable. If these boundaries can be assumed to form effectively infinite and fairly straight sections cutting through the aquifer, problems of flow toward wells located nearby become relatively easy to treat analytically. The method of images [6] affords a relatively simple procedure for solving problems of flow toward wells near such boundaries, as well as near others, such as [38, 39] infinite quadrants, infinite and semiinfinite strips, rectangles, and [10] sector aquifers, provided the wells are steadily discharging from horizontal aquifers of uniform thickness. The method becomes difficult to apply if the aquifer is otherwise; in such instances direct solution of the particular boundary-value problem generally becomes simpler.

A. The Method of Images

If a system of steadily discharging wells (sinks) and/or recharging wells (sources) exists in an infinite aquifer, the multiple-well potential is the sum of the single-well potentials [Section VI, A, 1, Eqs. (96) and (97)]. Furthermore, the space derivative of the multiple-well potential at a point in the flow system in any direction is the sum of the space derivatives of the single-well potentials at that point. Thus, if there are N wells and the well potential due to the ith well is Q_iZ_i, the multiple-well potential P and its space derivative $\partial P/\partial n$ in the direction n at any point in the flow system are, respectively, given by

$$P = \sum_{i=1}^{N} Q_i Z_i, \qquad \text{and} \qquad \partial P/\partial n = \sum_{i=1}^{N} Q_i \, \partial Z_i/\partial n$$

If a system of wells tapping a horizontal aquifer of uniform thickness is reflected negatively in a vertical and straight section of the flow system (that is, negative images $(-Q_i)$ are placed on the other side of the section), say the y-axis for convenience, the multiple-well potential of the new system, from the first of the preceding equations, is given by

$$P = \sum_{i=1}^{N} Q_i Z_i(R_i,t) + \sum_{i=1}^{N} (-Q_i) Z_i'(R_i',t) \tag{103}$$

in which

$$R_i^2 = (x-x_i)^2 + (y-y_i)^2 + z^2, \qquad R_i'^2 = (x+x_i)^2 + (y-y_i)^2 + z^2$$

and (x_i, y_i) and $(-x_i, y_i)$ are the coordinates of the center of the ith well and of its image, respectively. At any point in the yz-plane $(x = 0)$, $R_i = R_i'$. Consequently the multiple-well potential at any point in the yz-plane, from Eq. (103), is zero. Thus, this flow system of a hydraulic boundary (the yz-plane) on which the head remains uniform may be taken to represent an actual semi-infinite aquifer bounded along the y-axis by a line of constant head and drained by the N real wells.

If, instead of being negatively reflected $(-Q_i)$, the above mentioned well system is positively $(+Q_i)$ reflected in the yz-plane, the multiple-well potential will be given by

$$P = \sum_{i=1}^{N} Q_i Z_i(R_i,t) + \sum_{i=1}^{N} Q_i Z_i'(R_i',t) \tag{104}$$

the derivative of which with respect to x is given by

$$\partial P/\partial x = \sum_{i=1}^{N} (x-x_i) Q_i \, \partial Z_i/\partial x + \sum_{i=1}^{N} (x+x_i) Q_i \, \partial Z_i'/\partial x$$

Obviously, the value of this derivative (proportional to v_x) vanishes at any point on the yz-plane. Consequently, such a flow system of an impermeable hydraulic boundary (the yz-plane) may be assumed to represent a semi-infinite aquifer in which a real impermeable boundary lies along the y-axis and is drained by the N real wells.

The method of images, therefore, affords a means by which it is possible to create a hydraulic flow system that is equivalent to an actual system in which a physical boundary is known to exist. To create hydraulic flow systems simulating actual systems having boundaries other than those discussed above, the process of reflection may have to proceed back and forth an infinite number of times, depending on the form of the boundary. By the method of images, an aquifer of finite extent can be transformed to one of infinite extent, so that the flow equations developed for an infinite aquifer can be applied to this substitute system.

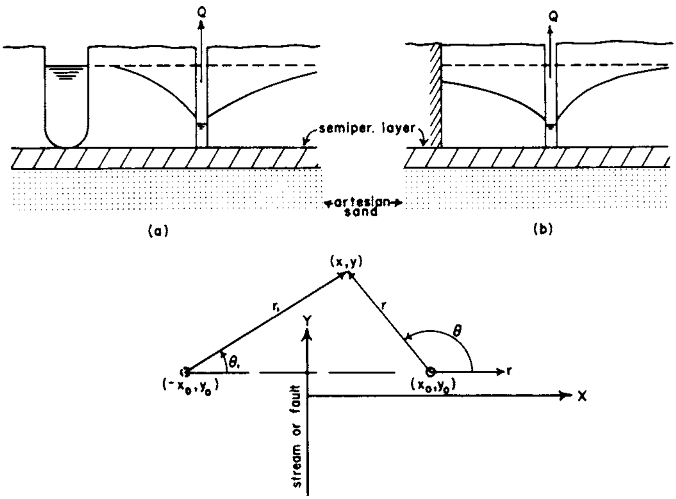

FIG. 18. Diagrammatic representation of a well in a leaky water-table aquifer (a) near a stream and (b) near an impermeable boundary.

B. Steady Wells near a Line of Constant Head

A line of constant head designates an effectively infinite and fairly straight section of an open boundary, such as banks of streams, lakes, and other water bodies, which cuts through the aquifer and on which the average head remains uniform.

1. *Wells in Horizontal Leaky Water-Table Aquifers*

Figure 18a represents diagrammatically a well near a stream, located along the y-axis, in a horizontal leaky water-table aquifer.

a. WATER-LEVEL DISTRIBUTION. If the well potential of Eq. (94) with $1/\beta = 0$ is used in conjunction with Eq. (103), the water-level distribution around a well located at (x_0,y_0) is given by

$$D_0^2 - \bar{D}^2 = (Q/2\pi K)\{W(u, r/B) - W(u_1,r_1/B)\} \qquad (105)$$

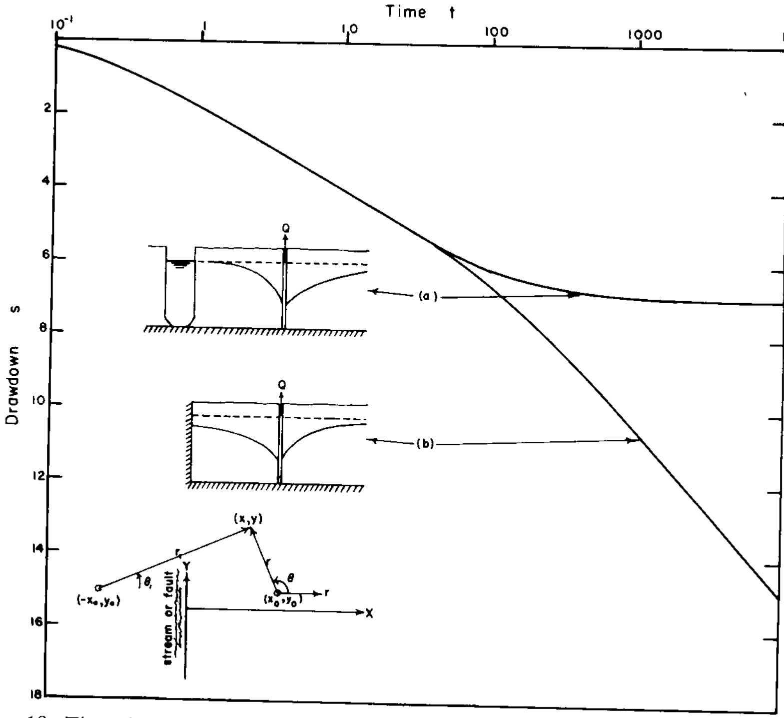

FIG. 19. Time-drawdown variation due to a steady well in a nonleaky water-table aquifer (a) near a stream and (b) near an impermeable boundary.

in which $r^2 = (x - x_0)^2 + (y - y_0)^2$, $r_1^2 = (x + x_0)^2 + (y - y_0)^2$, and $u_1 = r_1^2/4\nu t$.

The corresponding steady-state distribution, from Eq. (105) with $u = u_1 = 0$, is given by

$$D_0^2 - \bar{D}^2 = (Q/\pi K)\{K_0(r/B) - K_0(r_1/B)\}$$

b. RATE AND TOTAL VOLUME OF STREAM DEPLETION. The rate q_i and the total volume V_i of river depletion at the end of a continuous period of pumping t are given [40], respectively, by

$$q_i = (Q/2)\{\exp(-x_0/B)\ \mathrm{erfc}[U_0 - (x_0/2B)/U_0] + \exp(x_0/B)\ \mathrm{erfc}[U_0 + (x_0/2B)/U_0]\} \quad (106)$$

and

$$V_i = Qt\{(q_i/Q) + [U_0^2/(x_0/B)]\,[\exp(x_0/B)\ \mathrm{erfc}\ [U_0 + (x_0/2B)/U_0] - \exp(-x_0/B)\,\mathrm{erfc}[U_0 - (x_0/2B)/U_0]]\} \quad (107)$$

in which

$$U_0 = x_0/\sqrt{4\nu t}$$

During the steady state, which is effectively realized for $\nu t/B > 1 + \sqrt{1 + 0.5(x_0/B)}$ and $B \neq \infty$, Eqs. (106) and (107) yield

$$q_i = Q \exp(-x_0/B) \qquad \text{and} \qquad V_i = Q(t - x_0 B/2\nu) \exp(-x_0/B)$$

The corresponding expressions for the rate and that part of total well flow that is derived from storage, q_s and V_s, respectively, are given by

$$q_s = Q \exp[-(x_0/2BU_0)^2] \operatorname{erf}(U_0) \tag{108}$$

and

$$V_s = Qt[1 - (q_i + q_s)/Q]/(x_0/2BU_0)^2 \tag{109}$$

which, for $(x_0/2BU_0) = \sqrt{\nu t}/B < 0.3$, may be approximated by

$$V_s = Qt\{1 - 4i^2 \operatorname{erfc}(U_0)\} \tag{110}$$

The relations during the steady state are given from Eqs. (108) and (109) (as $t \to \infty$), effectively realized for $\nu t/B > 1 + \sqrt{1 + 0.5(x_0/B)}$, by

$$q_s = 0 \qquad \text{and} \qquad V_s = (B^2 Q/\nu)[1 - \exp(-x_0/B)]$$

The rate and volume of that part of total well flow that is from leakage are given by $q_L = Q - q_i - q_s$ and $V_L = Qt - V_i - V_s$, respectively.

2. *Wells in Horizontal Nonleaky Water-Table Aquifers*

The flow formulas around such wells are obtained from the limit, as $1/B \to 0$, of Eqs. (105) to (109).

a. WATER-LEVEL DISTRIBUTION. The unsteady water-level formula is

$$D_0^2 - \bar{D}^2 = (Q/2\pi K)\{W(u) - W(u_1)\}$$

and the corresponding steady-state formula is

$$D_0^2 - \bar{D}^2 = (Q/\pi K) \ln(r_1/r).$$

b. RATE AND VOLUME OF STREAM DEPLETION. The required expressions are

$$q_i = Q \operatorname{erfc}(U_0), \qquad q_s = Q \operatorname{erf}(U_0)$$

$$V_i = Qt[4i^2 \operatorname{erfc}(U_0)], \qquad \text{and} \qquad V_s = Qt[1 - 4i^2 \operatorname{erfc}(U_0)]$$

Example 23. A municipal well is pumped continuously for a period of 290 days (25×10^6 sec). The well is located near a fairly long and straight section of a river that cuts through a horizontal leaky water-table aquifer for which $T = 0.1$ ft²/sec, $\epsilon = 0.1$, and the leakage coefficient $K'/b' = 10^{-9}$ sec⁻¹. The effective distance of the well from the river is $x_0 = 10{,}000$ ft. Estimate the rate and the part of total volume pumped that is derived from river depletion, from storage, and from leakage.

For $\nu = T/\epsilon = 1$ ft²/sec, $x_0 = 10{,}000$ ft, $B = \sqrt{T/(K'/b')} = 10{,}000$ ft, $t = 25 \times 10^6$ sec, and $U_0 = 1$, Eqs. (106) to (108) and (110) give, respectively, $q_i/Q \approx 0.13$, $V_i/Qt \approx 0.05$, $q_s/Q \approx 0.67$, and $V_s/Qt \approx 0.80$. Consequently, that from leakage, $q_L/Q \approx 0.20$ and $V_L/Qt \approx 0.15$.

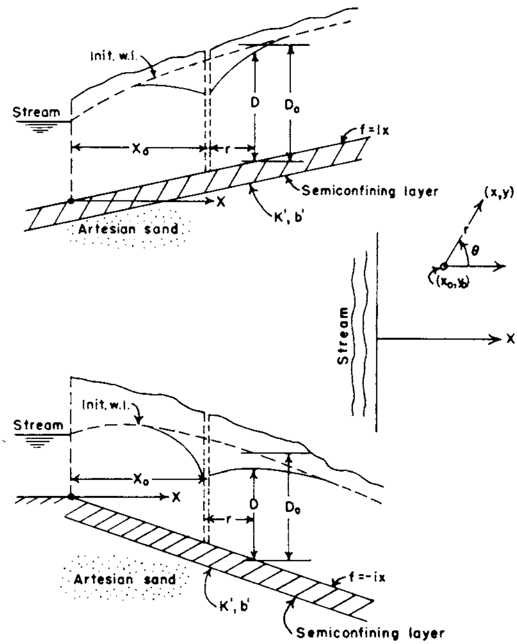

FIG. 20. Diagrammatic representation of a well near a stream in a sloping leaky water-table aquifer.

3. *Wells in Sloping Water-Table Aquifers*

Figure 20 illustrates situations of wells near a line of constant head in sloping leaky water-table aquifers.

The method of images can be used to create a hydraulic system in an infinite aquifer that is equivalent to the actual system of Fig. 20. The procedure, however, is not as straightforward as it is in the case of horizontal aquifers. The discharge of each of the negative image wells (strength of sources) must be adjusted by a constant factor (depending on the location of the real well) in order for the multiple-well potential on the plane of reflection to be equal to zero. By writing Eq. (103), using the single-well potentials from Eq. (94) and replacing $(-Q_i)$ with $(-C_iQ_i)$, evaluating at $x = 0$, and solving for C_i the value of C_i

will be found as equal to $\exp(2x_i/\beta)$. Thus, the water-level distribution around a well near a line of constant head in a leaky water-table aquifer, sloping upward in the direction of positive x (Fig. 20a), is given by

$$D_0^2 - \bar{D}^2 = (Q/2\pi K)\exp[-(x-x_0)/\beta]\{W(u,\alpha r) - W(u_1,\alpha r_1)\} \quad (111)$$

and the corresponding equation during the steady state ($t = \infty$, or $u = 0$) is

$$D_0^2 - \bar{D}^2 = (Q/\pi K)\exp[-(x-x_0)/\beta]\{K_0(\alpha r) - K_0(\alpha r_1)\}$$

For an aquifer sloping downward in the direction of positive x (Fig. 20b), the numerical value of β is negative. Consequently, the preceding equations pertain to this flow system after replacing β by $-\beta$.

C. Steady Wells near an Infinite Impermeable Barrier

An effectively infinite and fairly straight section of a closed boundary, such as a tight fault and similar structures, may be regarded as an infinite impermeable barrier across which flow does not take place. Figure 18b illustrates such a boundary in a horizontal water-table aquifer.

If the well potential of Eq. (94) with $1/\beta = 0$ is used in conjunction with Eq. (104), the water-level distribution around a well located at (x_0,y_0) near an impermeable barrier in horizontal water-table aquifers will be given by the corresponding equations of Section VII, B, 1 and 2, after replacing the negative sign between the braced terms with a positive sign. There will be no steady-state expression for the nonleaky case.

For a barrier along the yz-plane in a sloping water-table aquifer, the condition $\partial\varphi(0,y,z,t)/\partial x = 0$ must be satisfied, φ being the hydraulic head. Because such a condition is not satisfied by the application of the method of images (if at all possible), direct solutions of boundary-value problems describing the flow in such systems may prove to be a simpler procedure. Treatment of the distribution of water levels around wells in such systems is still pending.

D. Steady Artesian Wells near a Line of Constant Head or an Infinite Impermeable Barrier

The drawdown distribution around steadily discharging artesian wells near a line of constant head or near an impermeable barrier is obtained by using Eq. (103) or (104), respectively, in conjunction with the appropriate artesian well potential, the drawdown s of the appropriate drawdown equation replacing P, and the right-hand side of that drawdown equation replacing Q_iZ_i. For example, the drawdown around a partially penetrating well near an impermeable boundary along the y-axis in a leaky artesian aquifer without storage in the semipervious layer, from Eqs. (73) and (104), is given by

$$s = (Q/4\pi T)\{W(u_r,r/B_r) + f + \text{(same terms with } r_1 \text{ replacing } r)\}$$

and that around a well near a line of constant head in a nonleaky aquifer, from Eqs. (63) and (103), is given by

$$s = (Q/4\pi T)\{W(u) - W(u_1)\}$$

Equations (106) to (110), relating to river depletion and the corresponding equations for nonleaky aquifers, are applicable for the corresponding artesian systems provided the parameters of the water-table aquifer are replaced by their artesian counterparts.

In the above treatment of wells near infinite boundaries, the value of the parameter x_0 is the effective distance from the center of the well to the physical location of such boundaries. In practice this distance does not necessarily extend exactly to the actual location of the boundary. It may be somewhat in excess of the actual distance to the boundary site, especially if the boundary is sloping or semipervious or both. It can be chosen empirically, making allowances for these factors or can be approximately determined through aquifer-test techniques [41–43].

VIII. Drainage Wells

Surface irrigation, regardless of the method of its application, always results in some percolation of the applied water below the root zone of the farmed area. In fact, this percolation is essential in order to maintain a favorable salt status in the soil in semiarid and arid regions. In certain cases a substantial quantity of irrigation water must be "wasted" by liberal application as a necessary means of preventing increased salinity of the soil. The water-table aquifer underlying such farmed areas often rests on a semipervious layer through which leakage is supplied from, or discharged into, an underlying formation in which the piezometric head remains more or less uniform. In many instances drainage wells are used to remove such excess water and to maintain the water table at a preassigned depth below ground surface without exceeding desirable pumping lifts. The use of wells as drainage devices is not restricted to these situations only [44].

The problem of estimating the discharge and the spacing of a group of equally discharging identical wells, that would bring the water table to a desirable level at the end of a specified period of time and maintain it thereafter below this elevation, is a problem of well interference and may thus be solved by using appropriate interference equations. This procedure is laborious and may be time consuming. Advantage may be taken, however, of the fact that when a large number of equally spaced identical wells located on a regular grid pattern and pumping from a vertically replenished aquifer (from deep percolation and/or vertical leakage), the discharge of each of the wells (after a period of transient flow) is effectively obtained from vertical replenishment over an area surrounding the well and enclosed by what may be considered as a regular polygon of ground-water divide. In other words, the flow toward the well may be approximated by that which would take place in a closed circular aquifer of radius equal to that of the circle circumscribing the regular polygon of ground-water divide, the well being at the center of this closed aquifer and the discharge of the well during the steady state being sustained by vertical replenishment within this circular aquifer. For a given well spacing, the actual duty of the well during the steady state (that is, the rate at which the well is to be pumped, such that the water table is maintained at a given depth at the corners of the regular polygon of the ground-water divide) is somewhat less than that implicitly assumed in the above approximation, since the actual area of diversion is somewhat smaller than the assumed circle of diversion. Consequently, the spacing of wells as calculated from formulas based on this idealization of the flow system will be somewhat closer than actually needed.

A. Drainage Wells in Horizontal Leaky Water-Table Aquifers

In Fig. 21 a horizontal leaky water-table aquifer is receiving downward vertical percolation at a uniform rate w in volume per unit time per unit area. The aquifer is drained by steady wells located on a regular grid such that each well may be considered to drain a circular portion of the aquifer of a radius r_e, the flow across the outer boundary being zero.

1. *Initial Water Level*

If uniform deep percolation occurs continuously for a period τ_0 and ceases thereafter and if the water table underneath most of the irrigated area may be assumed to remain more or less horizontal (that is, the excess irrigation water added to the saturated permeable soil mantle is discharged mostly by vertical leakage through the underlying semipervious layer into the underlying constant-head artesian aquifer), the depth of saturation D_0 beneath the irrigated area, prior to operating the wells or after a complete recovery following a prolonged period of shut-down, may be shown to be approximated by

$$D_0 = h_u + [w/(K'/b')]F(\tau) \tag{112}$$

in which

$$F(\tau) = 1 - \exp(-\gamma\tau), \qquad \text{for} \qquad 0 < \tau < \tau_0$$
$$= \exp[-\gamma(\tau-\tau_0)] - \exp(-\gamma\tau), \qquad \text{for} \qquad \tau > \tau_0$$

and

$$\gamma = (K'/b')/\epsilon$$

where ϵ is the specific yield of the aquifer and K'/b' is the coefficient of leakage, w is the rate of uniform deep percolation, τ is the time since the incidence of deep percolation, τ_0 is the interval during which deep percolation occurs, and

h_u is the height (above the base of the aquifer) of the constant piezometric head in the underlying artesian aquifer, which, prior to the incidence of deep percolation or after a long period since cessation of deep percolation, coincides with the water table.

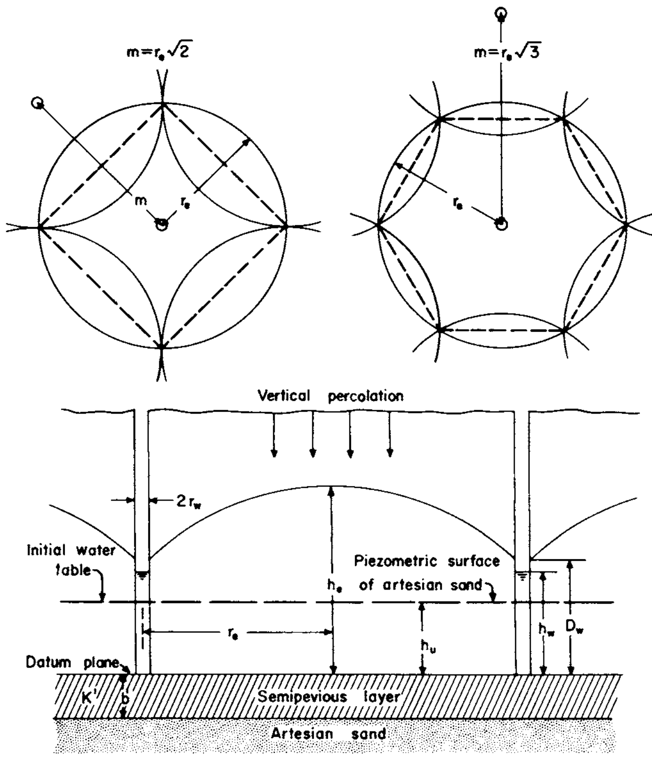

FIG. 21. Diagrammatic representation of drainage wells in a leaky water-table aquifer.

2. *Water Levels around a Steady Well*

If a steady well (a well of constant discharge) such as described in Fig. 21 begins to pump at time t^* since the incidence of deep percolation, the water levels around the well may be approximated by

$$D_0^2 - h^2 = (Q/\pi K)\{K_0(r/B) + K_1(r_e/B)I_0(r/B)/I_1(r_e/B) - 2(B/r_e)^2 \exp[-\gamma(\tau - t^*)] - 2\Sigma\} \tag{113}$$

in which

$$B^2 = [0.5D_0(t^*) + 0.25(h_e + D_w)]K/(K'/b') \tag{113a}$$

where h_e is the value of h at $r = r_e$, D_w is the height of the water table at the face of the well, $D_0(t^*)$ is the value of D_0 for $\tau = t^*$ and D_0 is given by Eq. (112);

Σ is an infinite series (see Eq. 12 of reference [45] for definition) that becomes of insignificant numerical value for $(\tau - t^*) > 3[\epsilon/(K'/b')]/[1 + 15(B/r_e)^2]$; h is the height of water table above the base of the water-table aquifer; Q is the discharge of the well; I_0, I_1, K_0, and K_1 are the Bessel functions defined in Section II, C; and τ is as defined for Eq. (112).

3. *Spacing and Discharge of Drainage Wells*

a. SPACING AND DISCHARGE EQUATIONS. Equation (113) may be used to estimate the discharge and the spacing of a grid of drainage wells that would bring the water table within a specified period of time t_0 and maintain it thereafter below a preassigned elevation without exceeding desirable pumping lifts, provided the time required to bring the water table to its new elevation is sufficiently large that it satisfies the criterion

$$t \geq 3[\epsilon/(K'/b')]/[1 + 15(B/r_e)^2] \tag{113b}$$

in which $t = \tau - t^*$ is the time since the well began to pump. At any time t, satisfying the preceding criterion, the water levels around the well may be approximated by Eq. (113) without the infinite series Σ. Evaluating Eq. (113), with $\Sigma = 0$ for $t = t_0$ at $r = r_e$ and at $r = r_w$, will yield, after mathematical reduction, the following relations:

$$Z(r_w/B, r_e/B) = [1 + f_1 - (h_w/D_0)^2]/[1 + f_1 - (h_e/D_0)^2] \tag{114}$$

and

$$Q = \pi K D_0^2[1 + f_1 - (h_e/D_0)^2](r_e/B)I_1(r_e/B) \tag{114a}$$

in which

$$f_1 = 2(Q/\pi K D_0^2)(B/r_e)^2 \exp(-\gamma t_0) \tag{114b}$$

$$Z = (r_e/B)[K_0(r_w/B)I_1(r_e/B) + K_1(r_e B)I_0(r_w/B)]$$

h_w is the height of the water level in the well (determined by the desired pumping lift), h_e is the value of h at $r = r_e$ (the desired maximum height of water table beneath the irrigated area), D_0 is given by Eq. (112) for $\tau = t^* + t_0$ where t_0 is the period of continuous pumping required to bring the water table to its new elevation, and r_w is the radius of the well.

Equation (114) cannot be explicitly solved; it can, however, be solved through use of a table of $Z(r_w/B, r_e/B)$ (Table IX) or through use of the following approximate relations, provided $r_w/r_e < 0.01$.

$$r_e/B = \alpha[1 + (\log_{10}\alpha)/2 \log_{10}(r_w/\alpha B)], \quad \text{if} \quad 0.03 < \alpha < 0.5 \tag{115}$$

and

$$r_e/B = \alpha[1 - 0.25\alpha^2], \quad \text{if} \quad 0.5 \leq \alpha < 1.5 \tag{115a}$$

in which

$$\alpha = 2\{(Z - 1)/[4.61 \log_{10}(B/r_w) - 1]\}^{0.5} \tag{115b}$$

and Z is the right-hand side of Eq. (114).

TABLE IX. Values of the Function $Z\left(\frac{r_w}{\beta}, \frac{r_e}{\beta}\right)$

r_w/β		0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90
	1	1.0437	1.1028	1.1889	1.3031	1.4460	1.6190	1.8227	2.0584	2.3278	2.6310	2.9705	3.3473	3.7626	4.2183	4.7167	5.2594	5.8475
	2	1.0402	1.0950	1.1750	1.2812	1.4145	1.5758	1.7661	1.9865	2.2384	2.5222	2.8400	3.1930	3.5821	4.0093	4.4767	4.9856	5.5375
	3	1.0381	1.0904	1.1669	1.2685	1.3960	1.5506	1.7330	1.9444	2.1861	2.4585	2.7637	3.1027	3.4766	3.8871	4.3363	4.8255	5.3560
	4	1.0367	1.0872	1.1611	1.2594	1.3829	1.5327	1.7095	1.9145	2.1490	2.4134	2.7096	3.0387	3.4017	3.8004	4.2367	4.7119	5.2274
10^{-5}	5	1.0356	1.0846	1.1566	1.2524	1.3728	1.5189	1.6913	1.8914	2.1202	2.3783	2.6676	2.9890	3.3435	3.7330	4.1593	4.6237	5.1275
	6	1.0347	1.0826	1.1529	1.2466	1.3645	1.5075	1.6765	1.8724	2.0967	2.3497	2.6333	2.9484	3.2961	3.6781	4.0962	4.5518	5.0459
	7	1.0339	1.0808	1.1498	1.2418	1.3575	1.4979	1.6639	1.8564	2.0768	2.3255	2.6042	2.9141	3.2560	3.6316	4.0428	4.4909	4.9769
	8	1.0332	1.0793	1.1471	1.2376	1.3514	1.4896	1.6530	1.8426	2.0596	2.3045	2.5791	2.8843	3.2212	3.5913	3.9966	4.4381	4.9172
	9	1.0326	1.0780	1.1448	1.2339	1.3460	1.4823	1.6434	1.8303	2.0444	2.2860	2.5569	2.8581	3.1906	3.5558	3.9558	4.3917	4.8645
	1	1.0321	1.0768	1.1427	1.2305	1.3412	1.4757	1.6348	1.8194	2.0308	2.2695	2.5371	2.8347	3.1631	3.5241	3.9193	4.3500	4.8174
	2	1.0286	1.0690	1.1287	1.2087	1.3097	1.4326	1.5782	1.7474	1.9415	2.1606	2.4066	2.6804	2.9827	3.3151	3.6793	4.0763	4.5073
	3	1.0266	1.0644	1.1206	1.1959	1.2912	1.4074	1.5451	1.7053	1.8892	2.0970	2.3303	2.5901	2.8771	3.1928	3.5388	3.9162	4.3259
	4	1.0252	1.0612	1.1148	1.1869	1.2781	1.3895	1.5216	1.6755	1.8521	2.0518	2.2762	2.5260	2.8022	3.1061	3.4392	3.8026	4.1971
10^{-4}	5	1.0241	1.0587	1.1103	1.1798	1.2680	1.3756	1.5034	1.6523	1.8183	2.0168	2.2402	2.4764	2.7441	3.0388	3.3619	3.7144	4.0973
	6	1.0231	1.0566	1.1066	1.1741	1.2597	1.3643	1.4886	1.6334	1.7998	1.9881	2.1999	2.4358	2.6967	2.9838	3.2988	3.6424	4.0157
	7	1.0224	1.0549	1.1035	1.1692	1.2527	1.3547	1.4760	1.6174	1.7799	1.9639	2.1708	2.4015	2.6565	2.9374	3.2454	3.5816	3.9468
	8	1.0217	1.0534	1.1009	1.1650	1.2466	1.3464	1.4651	1.6035	1.7627	1.9429	2.1457	2.3717	2.6218	2.8971	3.1991	3.5288	3.8870
	9	1.0211	1.0520	1.0985	1.1613	1.2412	1.3391	1.4555	1.5913	1.7475	1.9245	2.1235	2.3455	2.5911	2.8616	3.1584	3.4823	3.8343
	1	1.0206	1.0509	1.0964	1.1580	1.2364	1.3325	1.4469	1.5804	1.7339	1.9079	2.1037	2.3221	2.5637	2.8298	3.1219	3.4407	3.7872
	2	1.0171	1.0430	1.0824	1.1362	1.2049	1.2894	1.3903	1.5084	1.6446	1.7991	1.9732	2.1677	2.3832	2.6209	2.8818	3.1670	3.4771
	3	1.0151	1.0385	1.0743	1.1234	1.1864	1.2642	1.3572	1.4663	1.5923	1.7354	1.8969	2.0775	2.2777	2.4986	2.7414	3.0069	3.2957
	4	1.0136	1.0352	1.0685	1.1143	1.1734	1.2463	1.3338	1.4364	1.5552	1.6902	1.8428	2.0134	2.2028	2.4119	2.6418	2.8932	3.1670
10^{-3}	5	1.0125	1.0327	1.0640	1.1073	1.1632	1.2324	1.3155	1.4133	1.5264	1.6552	1.8008	1.9638	2.1447	2.3446	2.5645	2.8051	3.0672
	6	1.0116	1.0306	1.0604	1.1016	1.1549	1.2211	1.3007	1.3943	1.5029	1.6266	1.7665	1.9232	2.0972	2.2896	2.5014	2.7331	2.9856
	7	1.0108	1.0289	1.0573	1.0967	1.1479	1.2115	1.2881	1.3783	1.4830	1.6024	1.7375	1.8889	2.0571	2.2432	2.4480	2.6723	2.9167
	8	1.0102	1.0274	1.0546	1.0925	1.1418	1.2032	1.2769	1.3645	1.4658	1.5814	1.7123	1.8591	2.0224	2.2029	2.4017	2.6195	2.8569
	9	1.0096	1.0261	1.0522	1.0888	1.1365	1.1959	1.2676	1.3523	1.4506	1.5629	1.6902	1.8329	1.9917	2.1674	2.3610	2.5730	2.8042
10^{-2}	1	1.0090	1.0249	1.0501	1.0855	1.1317	1.1893	1.2590	1.3413	1.4370	1.5464	1.6703	1.8095	1.9643	2.1356	2.3245	2.5314	2.7571

r_w/β		0.90	0.95	1.00	1.20	1.40	1.60	1.80	2.00	2.20	2.40	2.60	2.80	3.00	3.20	3.40	3.60	3.80
	1	5.8475	6.4850	7.1746	10.4949	14.8752	20.5690	27.8962	37.2830	49.2043	64.3366	83.4675	107.6267	138.0278	176.2651	224.2669	284.4537	359.7649
	2	5.5375	6.1356	6.7828	9.9005	14.0154	19.3660	26.2531	35.0775	46.2858	60.5141	78.5029	101.2205	129.8081	165.7655	210.9054	267.5041	338.3259
	3	5.3560	5.9312	6.5536	9.5527	13.5124	18.6622	25.2918	33.7872	44.5784	58.2776	75.5983	97.4726	124.9993	159.6227	203.0881	257.5876	325.7829
	4	5.2274	5.7862	6.3911	9.3061	13.1556	18.1630	24.6100	32.8721	43.3673	56.6915	73.5382	94.8143	121.5887	155.2659	197.5438	250.5544	316.8869
10^{-5}	5	5.1275	5.6737	6.2649	9.1146	12.8786	17.7754	24.0807	32.1615	42.4271	55.4599	71.9387	92.7504	118.9406	151.8831	193.2390	245.0937	309.9798
	6	5.0459	5.5818	6.1619	8.9583	12.6526	17.4592	23.6487	31.5818	41.6599	54.4551	70.6336	91.0664	116.7798	149.1230	189.7265	240.6380	304.3440
	7	4.9769	5.5041	6.0747	8.8260	12.4613	17.1916	23.2832	31.0911	41.0106	53.6046	69.5291	89.6411	114.8319	146.7871	186.7539	236.8671	299.5743
	8	4.9172	5.4367	5.9992	8.7115	12.2955	16.9597	22.9665	30.6660	40.4480	52.8678	68.5721	88.4063	113.3668	144.7632	184.1783	233.5999	295.4418
	9	4.8645	5.3774	5.9327	8.6105	12.1495	16.7554	22.6875	30.2915	39.9524	52.2186	67.7290	87.3184	111.9710	142.9802	181.9093	230.7216	291.8011
	1	4.8174	5.3243	5.8731	8.5201	12.0187	16.5724	22.5419	29.9560	39.5084	51.6372	66.9738	86.3440	110.7207	141.3831	179.8769	228.1434	288.5399
	2	4.5073	4.9749	5.4814	7.9256	11.1589	15.3693	20.7944	27.7504	36.5897	47.8143	62.0088	79.9374	102.5007	130.8827	166.5144	211.1926	267.0995
	3	4.3259	4.7705	5.2522	7.5779	10.6559	14.6655	19.8332	26.4602	34.8824	45.5781	59.1045	76.1897	97.6923	124.7404	158.6978	201.2770	254.5577
	4	4.1971	4.6255	5.0896	7.3312	10.2990	14.1662	19.1512	25.5448	33.6710	43.9915	57.0438	73.5307	94.2807	120.3824	153.1519	194.2418	245.6591
10^{-4}	5	4.0973	4.5130	4.9635	7.1398	10.0222	13.7789	18.6222	24.8348	32.7314	42.7608	55.4454	71.4683	91.6344	117.0021	148.8501	188.7849	238.7569
	6	4.0157	4.4211	4.8604	6.9834	9.7960	13.4625	18.1900	24.2546	31.9637	41.7553	54.1394	69.7831	89.4723	114.2401	145.3354	184.3263	233.3669
	7	3.9468	4.3434	4.7733	6.8512	9.6048	13.1949	17.8246	23.7641	31.3146	40.9051	53.0353	68.3583	87.6442	111.9049	142.3636	180.5566	228.3491
	8	3.8870	4.2761	4.6978	6.7367	9.4391	12.9631	17.5080	23.3392	30.7523	40.1687	52.0788	67.1241	86.0607	109.8821	139.7894	177.2911	224.2187
	9	3.8343	4.2167	4.6313	6.6357	9.2930	12.7587	17.2288	22.9644	30.2564	39.5190	51.2351	66.0355	84.6639	108.0978	137.5188	174.4107	220.5754
	1	3.7872	4.1636	4.5717	6.5453	9.1623	12.5758	16.9790	22.6292	29.8127	38.9380	50.4804	65.0617	83.4144	106.5017	135.4877	171.8342	217.3165
	2	3.4771	3.8142	4.1799	5.9509	8.3024	11.3728	15.3359	20.4236	26.8941	35.1152	45.5154	58.6551	75.1944	96.0014	122.1253	154.8835	195.8762
	3	3.2957	3.6098	3.9508	5.6031	7.7994	10.6690	14.3747	19.1334	25.1867	32.8790	42.6111	54.9075	70.3861	89.8593	114.3089	144.9681	183.3346
	4	3.1670	3.4648	3.7882	5.3564	7.4426	10.1697	13.6927	18.2181	23.9754	31.2924	40.5506	52.2486	66.9746	85.5014	108.7632	137.9332	174.4363
10^{-3}	5	3.0672	3.3524	3.6621	5.1650	7.1658	9.7824	13.1638	17.5081	23.0359	30.0618	38.9523	50.1863	64.3285	82.1212	104.4617	132.4766	167.5345
	6	2.9856	3.2605	3.5590	5.0087	6.9396	9.4660	12.7316	16.9280	22.2682	29.0564	37.6464	48.5013	62.1666	79.3595	100.9472	128.0183	161.8954
	7	2.9167	3.1828	3.4719	4.8765	6.7484	9.1985	12.3662	16.4375	21.6192	28.2063	36.5424	47.0767	60.3387	77.0246	97.9758	124.2491	157.1278
	8	2.8569	3.1155	3.3965	4.7620	6.5828	8.9668	12.0497	16.0127	21.0570	27.4699	35.5860	45.8427	58.7554	75.0021	95.4020	120.9841	152.9980
	9	2.8042	3.0561	3.3299	4.6610	6.4367	8.7624	11.7706	15.6380	20.5611	26.8205	34.7425	44.7542	57.3589	73.2181	93.1318	118.1043	149.3555
10^{-2}	1	2.7571	3.0030	3.2704	4.5706	6.3060	8.5795	11.5209	15.3028	20.1176	26.2395	33.9880	43.7806	56.1097	71.6224	91.1012	115.5284	146.0972

The relative error in r_e as computed from Eq. (115) is at most (near $\alpha = 0.03$) 5% and that as computed from Eq. (115a) is at worst (near $\alpha = 1.5$) 10%; it is considerably less for most of the indicated ranges of α.

If $\alpha > 1.5$, resort should be made to the tabulation of $Z(r_w/B, r_e/B)$; see Table IX.

b. PROCEDURE FOR COMPUTATION. (1) The available data should include values of h_u, h_w, h_e, t^*, t_0, w, K'/b', ϵ, and K; (2) from these data and Eq. (112), compute D_0 for $\tau = t^* + t_0$; (3) compute B from Eq. (113a); if a value of D_w is not available, h_w may be used instead as a first approximation; (4) compute f_1 from Eq. (114b), with Q taken, as a first approximation, equal to $\pi w r_e^2$; (5) compute Z from Eq. (114); (6) compute α from Eq. (115b) and r_e from Eq. (115), Eq. (115a), or a table of $Z(r_w/B, r_e/B)$, whichever applies; (7) compute the first approximation of Q from Eq. (114a) and a table [15] of I_1; (8) with the values of Q and r_e thus obtained, f_1 is recomputed and steps (5) to (7) are repeated to obtain a second approximation of Q and r_e; (9) if the second approximation of Q differs significantly from its first approximation, steps (4) to (7) are repeated, using the second approximation values of r_e and Q to obtain a third approximation; usually, the second approximation is sufficiently accurate for practical use.

c. SPACING AND DISCHARGE EQUATIONS IN CASE OF ZERO DEEP PERCOLATION. If wells are used to drain an area waterlogged by upward leakage only, Eqs. (112) to (115b), with $w = 0$, will pertain to this drainage situation.

The procedure for computing r_e and Q is the same as outlined previously, using $f_1 = 0$ in the first step of the method of successive approximations as followed therein.

B. Drainage Wells in Horizontal Nonleaky Water-Table Aquifers

1. Equations for Water Levels

If the water-table aquifer of Fig. 21 rests on an impermeable base, the water level variation after the incidence of the uniform deep percolation w may be obtained, from Eq. (112) as $K' \to 0$, as

$$D_0 = h_0 + (w/\epsilon)F_1(\tau) \tag{116}$$

in which

$$F_1(\tau) = \tau \quad \text{for} \quad 0 < \tau < \tau_0$$
$$= \tau_0 \quad \text{for} \quad \tau > \tau_0$$

h_0 is the depth of saturation prior to the incidence of deep percolation; other symbols are as defined for Eq. (112).

The water levels around a steady well that begins to pump at time $\tau = t^*$ can be obtained, from Eq. (113) as $K' \to 0$, as

$$D_0^2 - h^2 = (Q/\pi K)\{2\nu(\tau - t^*)/r_e^2 + 0{\cdot}5(r/r_e)^2 - 0.75 + \ln(r_e/r) - 2\Sigma_1\} \quad (117)$$

in which

$$\nu = [0.5D_0(t^*) + 0.25(D_w + h_e)]K/\epsilon$$

where D_0 is given by Eq. (116) and $D_0(t^*)$ is the value of D_0 just before pumping started, that is the value of D_0 at $\tau = t^*$; and Σ_1 is the value of Σ with $1/B = 0$. Other symbols are as defined for Eq. (113).

In the process of obtaining Eq. (117) from Eq. (113), one should observe that as $x \to 0$ (in fact, for $x < 0.1$) the functions involved in Eq. (113) may be approximated as follows:

$$K_0(x) \simeq -0.5772 - \ln(x/2), \qquad I_0(x) \simeq 1 + 0.25x^2$$
$$K_1(x) \simeq -[0.5772 + \ln(x/2)]I_1(x) + 1/x - 0.25x$$
$$I_1(x) \simeq 0.5x(1 + 0.125x), \qquad \text{and} \qquad \exp(-x) \simeq 1 - x$$

2. *Spacing and Discharge Equations*

Provided that the time required to lower the water table to its new elevation satisfies the criterion

$$t = \tau - t^* \geq 0.4\epsilon r_e^2/K[D_0(t^*) + 0.5(D_w + h_e)] \quad (118)$$

a procedure similar to that leading to Eqs. (114) and (114a) will yield (after mathematical manipulation and neglecting Σ_1 in the process) the following relations:

$$f = \alpha^2 (\ln \alpha - 0.5 + 0.25\beta) \quad (119)$$

and

$$Q = \pi K D_0^2[1 - (h_e/D_0)^2]/[(\bar{t}/\alpha^2) - 0.25] \quad (119a)$$

in which

$$\alpha = r_e/r_w, \qquad \bar{t} = 2\nu(\tau - t^*)/r_w^2, \qquad f = \beta\bar{t}$$
$$\beta = [1 - (h_w/h_e)^2]/[(D_0/h_e)^2 - 1] \quad (119b)$$

and other symbols are as defined for Eqs. (116) and (117).

Equation (119) cannot be explicitly solved for α. However, if α and β satisfy the criterion $300 \leq \alpha \leq 3000$ and $\beta \leq 10$, within which practical values of α and β lie, Eq. (119) may be approximated, with errors not exceeding 1.5%, by the following relation:

$$\log_{10}(10^{-3}\alpha) = [(0.464 + 0.018\beta)/(1 + 0.036\beta)] \cdot \log_{10}[7.092(10^{-8})f/(0.464 + 0.018\beta)] \quad (120)$$

Thus, with f and β known from available data and Eqs. (119b), the value of α, and hence r_e, can be computed from Eq. (120). Consequently, Q is obtained from Eq. (119a).

The values of r_e and Q, as obtained previously, pertain to the problem of bringing the water table to a desired elevation within a given period of time t_0 [t_0 satisfies the criterion (118)] without exceeding desirable pumping lifts. In other words, the discharge Q is not equal to $\pi w r_e^2$ (the rate received from deep percolation), since water from storage additional to that from percolation must be removed. Consequently, the water levels around the wells will continue to change, if the wells are continued at the design discharge. If, however, the discharge of the well is changed, at the end of the period t_0, to a discharge equal to $\pi w r_e^2$, the water levels will become essentially stable for $t' \geq$ the right-hand member of Eq. (118), t' being the time since the change of the well discharge. The steady-state water level at $r = r_e$ may be shown to be given by

$$h_s^2(r_e) = h_e^2(r_e, t^* + t_0) - (Q - Q_w)/4\pi K \tag{120a}$$

in which h_e is the depth of saturation at $r = r_e$ just before changing the design discharge Q to $Q_w = \pi w r_e^2$ and h_s is the depth of saturation at $r = r_e$ during the steady state.

The counterparts of Eqs. (119) and (119a) during a steady state of flow is the limit of these equations as $t \to \infty$. If $t \to \infty$, then $D_0 \to \infty$, $\beta \to 0$, and $f = \bar{t}\beta \to (K/w)(h_e/r_w)^2[1-(h_w/h_e)^2]$. With these data, Eqs. (119) and (119a) will, respectively, reduce to

$$f = \alpha^2(\ln \alpha - 0.5)$$

and (120b)

$$Q = \pi w r_e^2 \qquad \text{with} \qquad f = (K/w)(h_e/r_w)^2[1-(h_w/h_e)^2]$$

which relations are obtained by Peterson [44] in his method of drainage well design and which he solves by graphical means. Peterson's graphical solution of the first of the above relations may be replaced by Eq. (120), with $\beta = 0$ and f is as defined in Eq. (120b).

Example 24. A leaky water-table aquifer receives an average rate of deep percolation of 1.93×10^{-7} ft³/sec/ft² during 6 mo of an irrigation season. Prior to the incidence of deep percolation, the water table in the soil mantle and the piezometric surface in the underlying constant-head artesian aquifer stand at 55 ft above the base of the soil mantle. The parameters ϵ, K, K'/b' are estimated as 0.10, 5×10^{-4} ft/sec, and 1.93×10^{-8} sec⁻¹, respectively. The permeable soil mantle is 75 ft deep and the ground water is to be maintained not less than 25 ft below ground surface within a period of 100 days since the beginning of the irrigation season by using 12-in. wells located on an equilateral triangular grid. If the wells are to be operated with the beginning of the irrigation season; estimate the discharge and well spacing, (1) if the pumping lift (neglecting well losses) is not to exceed 60 ft, and (2) if the water-table aquifer is not leaky.

(1) Given: $D_0(t^*) = D_0(0) = h_u = 55$ ft; $h_e = 75-25 = 50$ ft; $h_w = 75-60 = 15$ ft; $t^* = 0$; $t_0 = 8.64 \times 10^6$ sec; $K = 5 \times 10^{-4}$ ft/sec; $\epsilon = 0.10$; $K'/b' = 1.93 \times 10^{-8}$ sec^{-1}, $w = 1.93 \times 10^{-7}$ ft/sec; and $r_w = 0.5$ ft. Slide-rule calculations follow: From Eq. (113a), $B^2 = 1.13 \times 10^6$ft^2, or $B = 1.063 \times 10^3$ft; $\gamma = (K'/b')/\epsilon = 1.93 \times 10^{-7}$ sec; from Eq. (112), $D_0(t^* + t_0) = 63.2$ ft; from Eq. (114b), with $Q = \pi w r_e^2$, $f_1 \simeq 0.0413$; from Eq. (114), $Z = 2.39$; from Eq. (115b), $\alpha = 0.624$; from Eq. (115a), $r_e/B = 0.61$ and hence $r_e \simeq 650$ ft; from Eq. (114a) and a table of I_1, $Q \simeq 0.502$ ft^3/sec. A second trial with these values of Q and r_e gives $f_1 = 0.0815$. Repeating the computations with this value of f_1 gives $r_e = 620$ ft and $Q = 0.528$ ft^3/sec. For full coverage on an equilateral grid, the spacing parameter is $m = r_e\sqrt{3} \simeq 1075$ ft. If more accuracy is desired, the calculation is to be repeated with a value of f_1 based on the second approximation of r_e and Q. The time criterion of Eq. (113b) gives $t \geq 4$ days. Since $t_0 = 100$ days, the time criterion is satisfied.

(2) $D_0(t^*) = D_0(0) = h_0 = 55$; from Eq. (116), $D_0(t^* + t_0) = 71.7$ ft; from Eq. (119b), $\bar{t} = 1.51 \times 10^7$, $\beta = 0.86$, and $f = 1.3 \times 10^7$; from Eq. (120), $\alpha = 1350$, from which $r_e = 675$ ft; the spacing for an equilateral grid is $m = r_e\sqrt{3} = 1170$ ft; and from Eq. (119a), $Q = 0.515$ ft^3/sec.

If the discharge of the well is reduced to $Q_w = \pi w r_e^2 = 0.276$ ft^3/sec, the water level at $r = r_e$ will, from Eq. (120a), stand at about 49.99 ft above the base of the soil mantle after about 4.85 days [from Eq. (118)] since the change of the well discharge.

IX. Flow to Collector Wells

Horizontal wells, generally known as *collector wells*, have been used under favorable hydrological conditions as a means for ground-water recovery. They have been reported to yield large quantities of water when located adjacent to streams, or in permeable aquifers removed from surface-water supplies [5].

The yield of a collector well is generally estimated by using the Dupuit-Forchheimer well-discharge formula with an experimentally suggested equivalent-well radius [46]. That is, the collector well is replaced by a hypothetical vertical well that completely penetrates the aquifer and is of such a radius that the drawdown therein is equal to that which obtains in the collector well. The suggested value for the equivalent-well radius is approximately 75% of the average lateral length in the collector-well system [47]. The flow pattern around a collector well is, however, extremely complex. The thickness of the

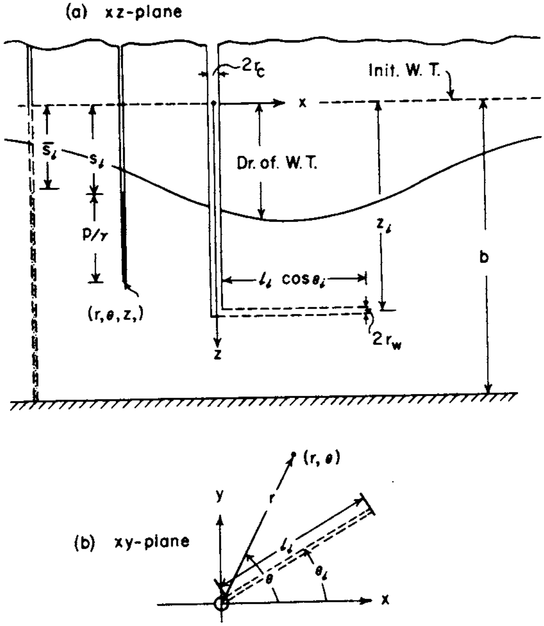

FIG. 22. Diagrammatic representation of a collector well in a nonleaky water-table aquifer.

aquifer and the length, location, and number of laterals are important factors in the determination of the yield of, and the drawdown distribution around, such wells.

Solutions for problems of flow toward steadily discharging, partially penetrating vertical wells have been obtained by treating the well as a line source, the strength of which (discharge) is uniformly distributed along the water-entry portion of the well [7, 22, 48–50]. These solutions have proved successful in practice, although, theoretically, the hydraulic head rather than the flux should be uniform along the face of the well. Collector wells are horizontally laid pipes of small diameters when compared with the aquifer thickness. They are partially penetrating, horizontal wells. Consequently, solutions for problems of flow toward collector wells obtained by treating the laterals as line sinks will probably be as successful in practical applications as their counterparts for vertical wells. Based on this assumption and on the usual assumptions with regard to the homogeneity and uniformity of the hydraulic properties of the aquifers (Section III), analytical solutions have been obtained [19] that describe the flow toward collector wells that are steadily discharging from water-table or artesian aquifers infinite in areal extent, or that are cut by fairly straight and effectively long stream channels. Some of the results are presented below.

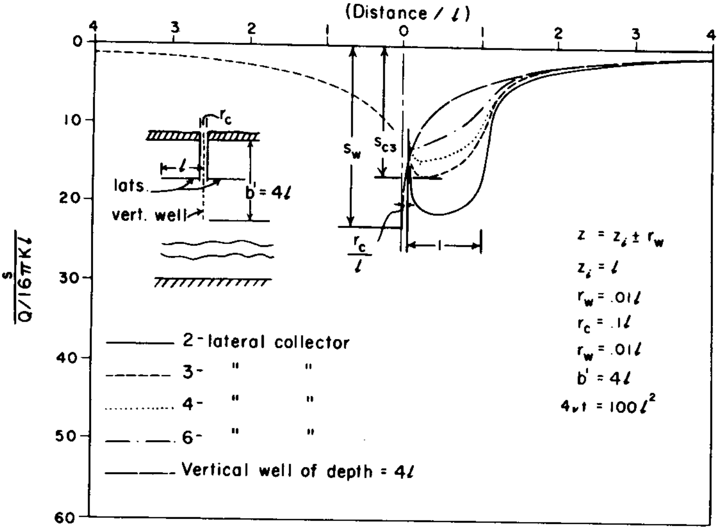

FIG. 23. Distance-drawdown variation in the plane of maximum drawdown due to the collectors shown in a thick aquifer.

A. Drawdown around Collector Wells

If each of a group of N laterals of a collector well is replaced by a finite line sink of uniform discharge along its axis and if the piezometric drawdown induced by the ith lateral at any point (r, θ, z) in the aquifer is denoted by s_i, the drawdown distribution s around the collector well is, by superposition, given by

$$s = \sum_{i=1}^{N} s_i = \sum_{i=1}^{N} (Q_i/l_i) f_i(r, \theta, z, t;\ \theta_i, z_i) \tag{121}$$

in which f_i is a function satisfying the boundary-value problem that governs the flow in a given flow system, and Q_i, l_i, z_i, and θ_i are the discharge, the length, the vertical position, and the orientation, respectively, of the ith lateral (see Fig. 22, for coordinate system). The discharge Q of the collector well is the sum of the discharges of the N laterals of the well.

1. Collector Well of Symmetrically Located Laterals

A group of radial laterals lying in the same horizontal plane and issuing from a central caisson are called *symmetrically located laterals*, provided that they are equal in length and are so oriented that each will drain an equal sector of an infinite, homogeneous, isotropic, and horizontal aquifer having a uniform thickness (Figs. 25a and 25b). When all are in operation, the discharge of each of the laterals of such a collector well is the same and is equal to Q/N. The drawdown around such wells is, therefore, given by

$$s = (Q/Nl)\sum_{i=1}^{N} f_i(r, \theta, z, t; \theta_i, z_1) \tag{122}$$

in which l is the length of each of the laterals and z_1 is the vertical coordinate of the laterals position.

2. *Pumping Levels in Collector Wells*

The equations of drawdown to be presented are based on the assumption that the flux entering each of the laterals is uniformly distributed along its water-entry face. Theoretically, the hydraulic head rather than the flux should be uniform along the face of the lateral. Because of field and operational conditions, neither a uniform flux nor a uniform head along the face of the lateral occurs in the actual field problem. In the field problem, the drawdown distribution along the face of the lateral will have a distribution between the two theoretical extremes. As in the problem of partially penetrating vertical wells [22], the drawdown in the collector well will be approximated by the maximum drawdown that obtains along the face of the uniformly discharging lateral.

The location of the point of maximum drawdown along the face of a lateral depends on the geometry of the collector well. In the case of a single lateral, this point occurs at the middle of the lateral. Computations in the drawdown equations show that this point approaches the face of the caisson of the collector well as the number of the laterals is increased (Fig. 23). For a well of four or more symmetrically located laterals, this point can, for all practical purposes, be taken along one of the laterals at the face of the caisson; that is $r_m = r_c$ provided $r_c < 0.05l$, r_c being the effective radius of the caisson and r_m the radial distance to the point of maximum drawdown. If the number of laterals in such wells is six or more, the same approximation can be made if $r_c < 0.1l$. The location of the point of maximum drawdown is independent of the hydraulic properties of the aquifer. Consequently, if $r_c > 0.1l$ (or if an exact location is required), the point can be located by constructing a drawdown profile along the face of one of the symmetrically located laterals (or if otherwise, along a centrally located lateral) by using assumed values of the hydraulic properties of the aquifer.

In the strict sense, the drawdown in the collector well is obtained by evaluating Eqs. (121) and (122) at the point of maximum drawdown. If $r_m = r_c < 0.1l$ (for four or more symmetrically located laterals, generally used in practice), however, the drawdown in a well of symmetrically located laterals may be closely approximated [19] by

$$\begin{aligned} s_c = (Q/Nl)\{&f_1(r_c, \theta_1, z_1 \pm r_w, t; \theta_1, z_1) \\ &+ (N-1)f_1(0, \theta_1, z_1 \pm r_w, t; \theta_1, z_1)\} \end{aligned} \tag{123}$$

in which θ_1, z_1, and r_w are, respectively, the angular coordinate (orientation), the vertical coordinate (the position), and the effective radius of the lateral along whose face the point of maximum drawdown lies (any of a group of symmetrically located laterals).

B. Collector Wells in Infinite Water-Table Aquifers

By assuming the maximum drawdown of the water table around a steadily discharging collector well is small relative to the original depth of flow (less than 25%), that the quantity of water released from storage because of compaction of the aquifer is small relative to that released by dewatering the aquifer, that the radius of the lateral is small relative to the depth of saturation (or thickness of the aquifer, if it is artesian), and that the radius of the caisson is small compared with the length of the lateral, the solutions presented herein [19] are obtained.

1. *Equations of Drawdown*

The equations given below are for the drawdown $s_i(r, \theta, z, t; \theta_i, z_i)$ induced by the ith of a group of laterals of a collector well. The total drawdown around the well is obtained through use of Eq. (121) or (122).

a. EQUATIONS FOR LONG TIMES. For t greater than both $2.5b^2/\nu$ and $5(r^2+l_i^2)/\nu$, the drawdown equation can be approximated (see Fig. 22 for the coordinate system) by

$$
\begin{aligned}
s_i = {} & [(Q_i/l_i)/4\pi Kb]\Big\{\alpha W[(\alpha^2+\beta^2)/4\nu t] - \delta W[(\delta^2+\beta^2)/4\nu t] \\
& -2\beta[\tan^{-1}(\alpha/\beta) - \tan^{-1}(\delta/\beta)] + 2l_i \\
& + (4b/\pi)\sum_{n=1}^{\infty}(1/n)[L(n\pi\alpha/b, n\pi\beta/b) - L(n\pi\delta/b, n\pi\beta/b)] \\
& \cdot \cos(n\pi z/b)\cos(n\pi z_i/b)\Big\}
\end{aligned}
\tag{124}
$$

in which

$$
\begin{aligned}
\alpha &= r\cos(\theta-\theta_i) - r_c, & \beta &= r\sin(\theta-\theta_i) \\
\delta &= r\cos(\theta-\theta_i) - l', & l' &= r_c + l_i \\
r^2 &= x^2+y^2, & \nu &= Kb/\epsilon
\end{aligned}
$$

the functions $W(u)$ and $L(u, \pm w)$ are defined in Section II, C, and $\tan^{-1}$ denotes the inverse circular tangent, tabular values for which are available.

b. DRAWDOWN IN COMPLETELY PENETRATING OBSERVATION WELLS. The average drawdown in an observation well screened throughout the aquifer may be obtained by integrating the piezometric drawdown, $s_i(r, \theta, z, t; \theta_i, z_i)$, with respect to z between the limits 0 and b and dividing the result by b (Section IV). For large times, the result is immediately seen to be given by Eq. (124) without the series terms.

Figure 24 compares drawdowns induced by a collector well in observation wells with drawdowns induced by a vertical well, located at the center of the caisson, and having a diameter equal to that of the symmetrically located laterals.

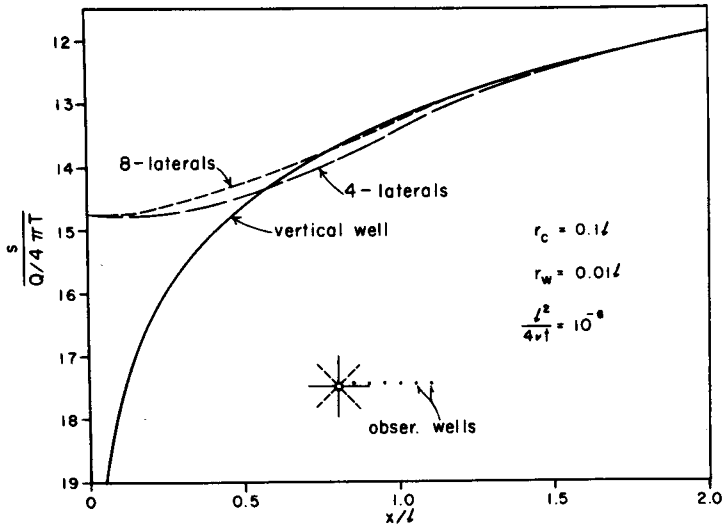

FIG. 24. Drawdown in observation wells due to the steady wells shown.

c. DRAWDOWN AT $r > l' + b$. In this region, the bracketed factor in the series of Eq. (124) approaches zero. Consequently, Eq. (124) without the series terms approximates the drawdown distribution in this region. If, in this region, $r > 5l'$, the equation of drawdown for a collector well of at least two laterals on a line can, from Eqs. (122) and (124) without the series terms, be given by the Theis formula; namely,

$$s = (Q/4\pi Kb)W(r^2/4\nu t)$$

Figures 25a and 25b give drawdown distributions around collector wells in the planes of their laterals. It is apparent that at large distances from the center of the caisson, the flow changes from three-dimensional in character to a radial type that can hardly be distinguished from that given by the preceding equation; that is, from that of a purely radial flow to a vertical well.

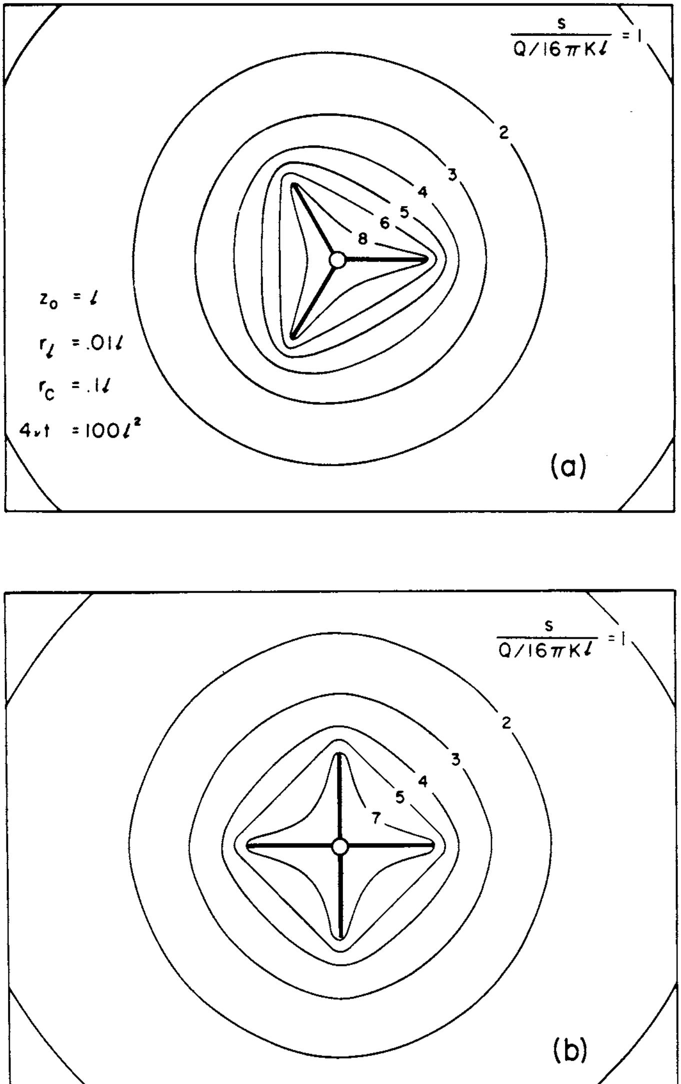

FIG. 25. Drawdown distribution in the plane of laterals of the collector wells shown in a thick aquifer.

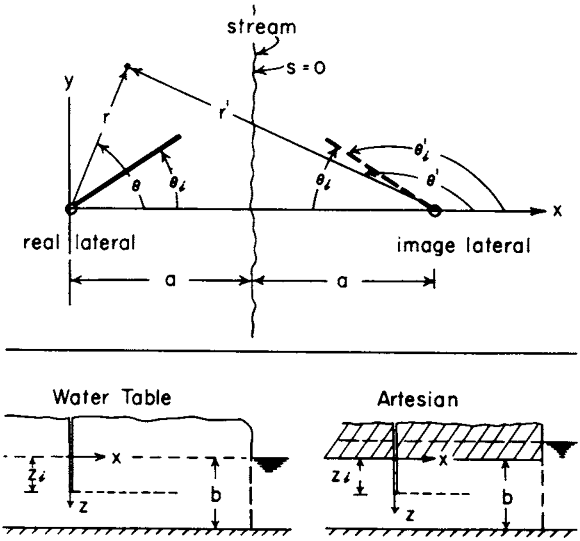

FIG. 26. Diagrammatic representation of a collector well near a stream.

2. *Equations for Pumping Level*

For $t > 2.5b^2/\nu$ and $>5(r_c^2 + l^2)/\nu$, the pumping levels in wells of four or more symmetrically located laterals can, if $l > 0.5b$ and $r_w < b/2\pi$ (for other situations and time criterion, see reference [19]), be approximated by the following relation:

$$\begin{aligned} s_c = (Q/4\pi KbN)\Big\{ & W(l^2/4\nu t) \\ & + [(N-1)/l][l' W(l'^2/4\nu t) - r_c W(r_c^2/4\nu t)] \\ & + 2N + (b/2l)\ln[(b/\pi r_w)^2/2(1 - \cos\pi(2z_i + r_w)/b)] \\ & + [4b(N-1)/\pi l]\sum_{n=1}^{M'}(1/n)[\pi/2 - L(n\pi r_c/b, 0)] \\ & \cdot \cos(n\pi z_i/b)\cos n\pi(z_i + r_w)/b\Big\} \end{aligned} \tag{125}$$

in which M' is an integer large enough so that $M' > b/2r_c$.

If $r_c > 0.5b$, the series in Eq. (125), for all practical purposes, may be neglected.

The drawdown in a collector well is always less than that obtaining at $r = 0$, regardless of the number of laterals of the well. The difference between

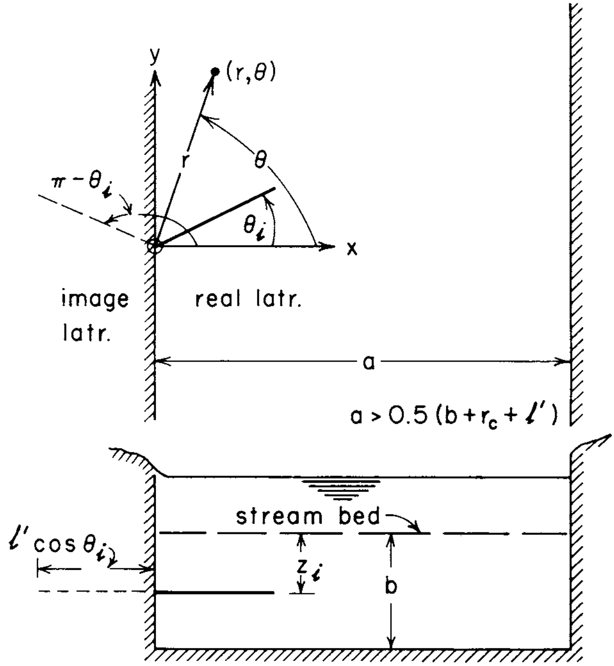

FIG. 27. Diagrammatic representation of a collector well under a stream bed.

these two values becomes increasingly smaller as the number of the laterals is increased or as the radius of the caisson becomes a small fraction of the depth of saturation (the thickness of the aquifer for the artesian case), or both. For $t > 5l^2/\nu$, $l > 0.5b$, and $r_w < b/2\pi$, the drawdown in the well is

$$s_c > (Q/2\pi Kb)\ \ln(R/l) \tag{126}$$

where

$$R = 4{\cdot}08\sqrt{\nu t}\,\{(b/\pi r_w)^2/2[1 - \cos \pi(2z_i + r_w)/b]\}^{b/4l} \tag{127}$$

C. Collector Wells near a Stream

Figure 26 represents a collector well near a stream of vanishing low grade and of a fairly straight and effectively long channel, that cuts completely through a water-bearing stratum. The distribution of drawdown around steadily discharging collector wells in such a system can be obtained by the use of the method of images (Section VII) and Eq. (124). The pumping level in the well during the steady state is of practical interest.

Pumping level during steady state: If $l > 0.5b$ and $r_w < b/2\pi$, the steady-state pumping level in a well having N symmetrically located laterals can be computed [19] from

$$s_c = (Q/2\pi KbN)\{\ln(\gamma^\gamma/\mu^\mu) - (N-1)\ln[\mu^\mu j^j/\gamma^\gamma \rho^\rho] + 0.5[\text{the logarithm and the series terms of Eq. (125)}]\} \quad (128)$$

in which $\gamma = 2(a-r_c)/l$, $\mu = (2a-2r_c-l)/l$, $j = l'/l$, and $\rho = r_c/l$, where a is the effective distance from the center of the collector well to the stream (Fig. 26). This distance, as in the case of vertical wells near a stream, does not necessarily extend exactly to the actual location of the stream bank (Section VII).

Provided that $l > 0.5b$ and $r_w < b/2\pi$, the drawdown in the well satisfies the following relation:

$$s_c > (Q/2\pi Kb)\ln R(\gamma^\gamma/\mu^\mu) \quad (129)$$

in which R is as given by Eq. (127).

D. Collector Wells under Stream Beds

Figure 27 shows a collector well under a stream bed. If the capacity of a stream channel is large compared to the maximum diversion of ground water, the slope of its surface may be neglected. If it is assumed that the percentage of the discharge of a collector well that originates from storage in the inland portion of the aquifer is small relative to that which originates from induced infiltration into the aquifer beneath the stream bed, it may be assumed that the banks of the effectively infinite and fairly straight stream are vertical impermeable planes that cut completely through the aquifer; hence, the method of images may be used in conjunction with Eq. (124) to obtain drawdown distribution around collector wells under stream beds.

Pumping level during steady state: The drawdown in the well during the steady state is of practical importance. This state of flow prevails for $t > 5b^2/\nu$. During this period, and provided that $a > 0.5(b + r_c + l')$ and that the laterals and their images form four or more symmetrically located laterals, the pumping level can be estimated from the following relation:

$$\begin{aligned} s_c/(Q/8\pi KlN) &= \ln \frac{[1-\cos\pi(2z_i+r_w)/2b][1+\cos\pi(r_w/2b)]}{[1+\cos\pi(2z_i+r_w)/2b][1-\cos\pi(r_w/2b)]} \\ &+ (16/\pi)\sum_{n=0}^{M'}[1/(2n+1)]\{L[(2n+1)\pi l/2b, 0] \\ &+ L[(2n+1)\pi(l'+r_c)/2b, 0] - L[(2n+1)\pi r_c/b, 0] \\ &- \pi/2 + 2(N-1)[L[(2n+1)\pi l'/2b, 0] \\ &- L[(2n+1)\pi r_c/2b, 0]]\}\sin[(2n+1)\pi(z_i+r_w)/2b] \\ &\cdot \sin[(2n+1)\pi z_i/2b] \end{aligned} \quad (130a)$$

which for $l > b$ and $r_w < b/\pi$ can be approximated by

$$s_c/(Q/8\pi KlN) = \ln\left\{(4b/\pi r_w)^2 \frac{[1 - \cos\pi(2z_i + r_w)/2b]}{[1 + \cos\pi(2z_i + r_w)/2b]}\right\}$$

$$+ (16/\pi)\sum_{n=0}^{M'} [1/(2n+1)]\{\pi/2 - L[(2n+1)\pi r_c/b, 0]$$

$$+ 2(N-1)[\pi/2 - L[(2n+1)\pi r_c/2b, 0]]\}$$

$$\cdot \sin[(2n+1)\pi(z_i + r_w)/2b] \sin[(2n+1)\pi z_i/2b] \qquad (130b)$$

in which M' is an integer so that $M' > 0.5b/r_c$.

Within the same limitations imposed on the preceding equation, the following also holds:

$$s_c > (Q/4\pi Kl) \ln\left\{(4b/\pi r_w)^2 \frac{[1 - \cos\pi(2z_i + r_w)/2b]}{[1 + \cos\pi(2z_i + r_w)/2b]}\right\} \qquad (131)$$

When a single lateral, normal to the stream, extends from bank to bank under a stream of width a, the drawdown in the caisson can be shown (from the comparable problem of a vertical well in an infinite strip of a nonleaky aquifer [38, p. 925, case 2]) to be given by

$$S_c = (Q/4\pi Ka)\ \{\text{the logarithmic term of Eq. (130a)}\}$$

At first glance, computations using these equations may appear laborious and difficult. However, inasmuch as the necessary mathematical tables for the functions involved are available, and inasmuch as only a few terms of the finite sums are required to obtain results accurate enough in practical applications, the computations are not so lengthy and difficult as they may at first appear.

Example 25. Estimate the upper limit of the yield of a collector well to be constructed under a river bed, if $l = 200$ ft, $b = 50$ ft, $a = 250$ ft, $z_i = 25$ ft, $r_w = 1$ ft, $r_c = 10$ ft, $K = 0.001$ ft/sec, and if the drawdown in the well is not to exceed 20 ft.

Since $a > 0.5(b + r_c + l')$, $l > b$, and $r_w > b/\pi$, the yield of the well will not exceed that given by Eq. (131). Thus, $Q < 4\pi(0.001)(200)(20)/2.3 \log_{10}[(200/\pi)^2(0.305/1.695)]$, or $Q < 7.62$ ft^3/sec.

X. Pumping Tests

Knowledge of the physical conditions of aquifers and of their hydraulic properties is essential in quantitative studies of ground-water resources. Determination of numerical values of the coefficients of storage, transmissivity, and leakage enables the evaluation of the importance of an aquifer as a fully developed source of water. These coefficients depend on factors characteristic of the aquifer and the water flowing through it. Because of the many factors on which these coefficients depend, numerical values of such coefficients must depend on experimental determination.

Although various laboratory techniques are available [5, 27], the numerical values obtained therefrom are only qualitatively significant for hydrologic purposes. More reliable values can be obtained from aquifer tests by which the water-bearing materials are tested under natural conditions. Such tests may be regarded as techniques for aquifer calibration. Several of these techniques are based on the theory of flow toward wells. They are generally referred to as pumping tests.

The ideal way of making a pumping test is to select a well at a considerable distance from any other pumped well, one that can be discharged at a large rate and shutdown at will; to install a large number of observation wells in all directions at varying distances from the well to be discharged; and to discharge and shutdown the well, frequently observing the water levels in all the wells so as to obtain representative drawdown and recovery data. Ordinarily, however, one of a group of available idle wells is pumped for as long as necessary or as long as operational conditions permit. The water levels in all available idle wells are observed during the entire period of drawdown and recovery.

There are no specific rules that can apply to all types of pumping tests. Each test is conducted under certain prevailing conditions (suggestions for conducting pumping tests may be found in references [27] and [51]) which are seldom, if ever, identical in any two tests. Because of the uncertainties and variable factors involved, there is no substitute for experience and judgement of qualified persons conducting these tests.

The analyses of data collected during a pumping test depend largely on the selection of the well-flow equation applicable to the flow system under investigation and on the degree to which the basic assumptions, governing the applicability of the flow equation, are in accord with the geologic characteristics of the aquifer being studied. Observed drawdown variations with time and distance from the pumped well may be interpreted in several ways (see Sections III, F; IV, F; and V, C, 2) if other supporting data are not available. Indiscriminate use and analyses of observed data may give erroneous and, in many instances, unreasonable results. This, of course, leads to the vexations that arise when attempts are made to force the application of formulas to situations where they do not apply. On the other hand, if the data are carefully studied and all possible interpretations are considered and screened, the analyses may provide valuable information not only about the hydraulic properties, but in many instances about the physical conditions and geometrical parameters of the flow system, such as the nature and location of hydraulic boundaries [51, 52], the average thickness of an aquifer [53], the effective distance to a line of constant head [41–43], the effective radii of wells and well losses [54, 55] and the tilt of a sloping water-table aquifer.

The method of analysis suited for any particular situation depends, among other factors, on the type of flow, the geometry of the flow system, the range of time within which the observed data fall, and the distribution in time and distance of the collected data. The methods outlined subsequently pertain to wells discharging at a constant rate from effectively infinite aquifers. Reference will be made to methods applicable to other flow systems and conditions of flow.

A. Methods of Analyses in Nonleaky Artesian Aquifers

Constructing semilogarithmic plot of observed drawdowns s against the time t (s being on the uniform scale) is an integral part of the process of conducting and analyzing pumping tests. Not only does it help in estimating the necessary length of a test, extending, shortening, or entirely changing the test, but also serves as a guide as to what method of analysis is most suited for a given situation. Thus, the first step in the application of any of the methods of analysis outlined subsequently is the preparation of such curves for all observation wells. They will, henceforth, be referred to as the "semilogarithmic data plots."

1. Type-Curve Methods

a. THEIS' TYPE-CURVE METHOD. This method constitutes the basis for all other "Type-Curve Methods," since the principle on which these methods are based is the same. It is named after C. V. Theis of United States Geological Survey who advanced the basic approach of solution. The following discussion pertains to the solution of the Theis formula. It applies, however, to other analogous equations provided the mathematical function involved in the equation of flow does not depend on more than two independent variables.

The Theis formula is given by

$$s = (Q/4\pi T)W(u) \tag{132}$$

with

$$u = r^2S/4Tt \tag{133}$$

For a given aquifer, T and S are presumably constants. If Q is constant, s will be proportional to the function $W(u)$. Similarly, the independent variable u of the function $W(u)$ will be proportional to r^2/t. Taking the common logarithm of Eqs. (132) and (133) will, therefore, yield

$$\log_{10}s - \log_{10}W(u) = \log_{10}(Q/4\pi T) = \text{constant}$$

and

$$\log_{10}(t/r^2) - \log_{10}(1/u) = \log_{10}(S/4T) = \text{constant};$$

that is, a logarithmic plot of $W(u)$ against $1/u$, called a *type curve*, is similar to that of s versus t/r^2, called a *data curve*, if the two plots are of the same scale. In other words, if the data curve is superposed on the segment of the type curve corresponding to the data curve, the s and $W(u)$ scales are displaced with respect to one another by the constant amount $\log_{10}(Q/4\pi T)$ and the t/r^2 and $1/u$ scales are displaced by the amount $\log_{10}(S/4T)$. This observation makes it possible to solve the drawdown equation (which otherwise cannot be solved explicitly) for the unknown values of T and S using pumping-test data collected from one or more observation wells. Data collected from the pumped well may be used only if the well losses are negligible or can be reasonably estimated by using equations of head loss in pipes and screens. The procedure follows:

(1) Construct a type curve of $W(u)$ against $1/u$ on a logarithmic paper.

(2) Plot observed values of s against t/r^2 (or against t, if one observation well is used) on a logarithmic paper to the same scale of the type curve.

(3) Superpose the data sheet on the type-curve sheet and translate vertically and/or horizontally, keeping the coordinate axes of the two plots parallel, to a position which gives the best fit of the observed points to the type curve.

(4) Select a point called a *matching point*, anywhere on the overlapping sheets and record its data coordinates (that is: s, t/r^2) as well as its type-curve coordinates [that is: $W(u)$, $1/u$], henceforth referred to as the *dual coordinates of the matching point*. For convenience, the matching point is selected such that one of its dual coordinates are both 1 or 0.1.

(5) Substitution of the dual coordinates in Eq. (132), then Eq. (133), will solve for T and S successively.

Application of the method is illustrated in Fig. 28 where field data from two observation wells are used.

The Theis type-curve method may be used to analyze recovery data, if the time-drawdown variation had the well continued to discharge could be extrapolated by extending the semilogarithmic time-drawdown curve through the period of recovery. In such a case, the data curve will be a logarithmic plot of $s-s'$ against t'/r^2, where t' is the time since shutdown, s is the extrapolated drawdown had pumping continued, and s' is the drawdown during recovery, or what is referred to as *residual drawdown*.

The Theis type-curve method, and consequently all type-curve methods, is a suitable method of analysis if the logarithmic plot of most of the observed points exhibits a well defined curvature (such as that of the type curve in the range $1/u < 100$) if a reasonably unique matching position is to be assured. The points falling within the period before the attainment of the straight line variation of the semilogarithmic data curve will, theoretically, exhibit such a curvature on the logarithmic data plot. It should be remembered, however, that in general less weight should be given to the early part of the data, since these data may not be closely represented by the theoretical drawdown equation on which the type curve (or curves) is based. The theoretical equations are based, among other things, on the assumptions that the well discharge remains constant and the release of the water stored in the aquifer is immediate and directly proportional to the rate of decline of the pressure head. Actually, there may be a time-lag between the pressure decline and the release of stored water and, initially also, the well discharge may vary as the pump is adjusting itself to the changing head, causing probable initial disagreement between theory and actual flow conditions. As the time of pumping becomes large, the effects of such flow conditions are minimized and closer agreement may be attained.

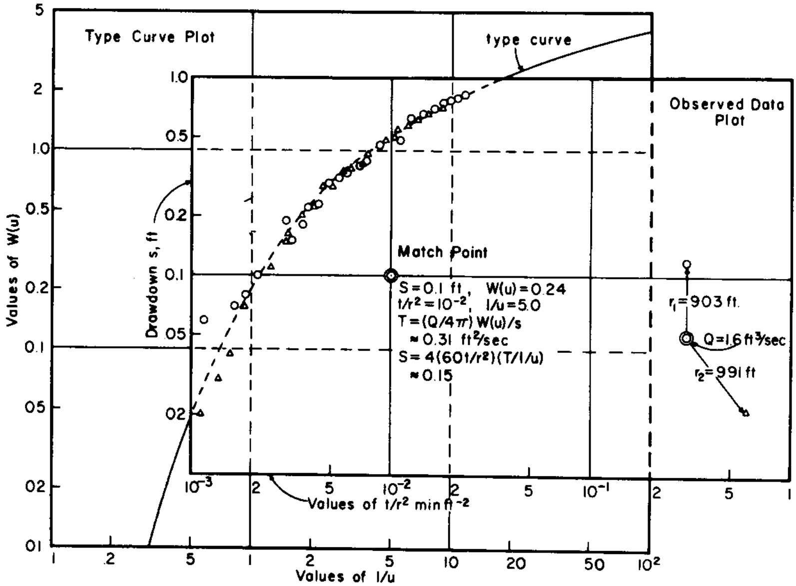

FIG. 28. Example of Theis type-curve method.

If the observed data on the logarithmic plot exhibits a flat curvature (such as that of the type curve for $1/u > 100$), several apparently reasonably good matching positions (depending on judgement) may be obtained; whence the graphical solution becomes practically indeterminate. In such cases, resort to other methods must be made.

b. HANTUSH PARTIAL PENETRATION TYPE-CURVE METHOD. Data collected from observation wells, located at $r > 1.5b$ from a partially penetrating pumping well or from wells screened throughout the aquifer, can be analyzed by the Theis type-curve method, since the average drawdown in such wells is described by the Theis formula (Section IV, D, 4). If such wells are not available, however, a type-curve method based on Eq. (82) may be used if the aquifer is isotropic and relatively deep. The method determines the hydraulic properties and often the average thickness of the aquifer. For detailed account of this method as well as others, the reader is referred to [53].

If a well pumping from an isotropic aquifer of thickness b is screened between the depths d and l, the average drawdown in observation wells screened between d' and l' may for all practical purposes be given by

$$\bar{s} = [Q/8\pi K(l-d)]E(u) \tag{134}$$

with

$$u = r^2S/4Tt = r^2Ss/4Kt \tag{135}$$

provided

$$t < S[2b - 0.5(2l + l' + d')]^2/20T$$

in which the function $E(u)$ is that given by Eq. (82), with $a = 1$ and $z = 0.5(l' + d')$.

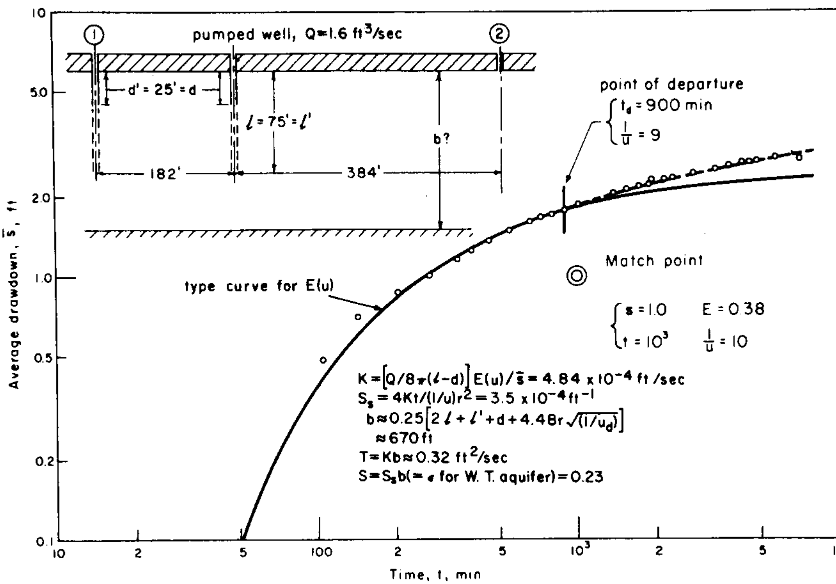

FIG. 29. Example of Hantush type-curve method for partial penetration.

In this method, a type curve for each observation well is necessary. The type curve is a logarithmic plot of $E(u)$ against $1/u$ and the data curve is that of $\bar{s}$ against t. The procedure of solution is the same as that of the Theis method. In matching the data curve to the type curve, however, one should remember that the observed points corresponding to relatively large values of time may deviate upward (having larger drawdown values) from the type curve. The deviation is to be expected, since the type curve represents drawdowns during relatively short period of pumping. Should, in the matching position, the data curve depart from the type curve, the value of u (designated by u_d) at the point where the two curves begin to depart from each other, called the *point of departure*, should be recorded in addition to the dual coordinates of a selected matching point. The dual coordinates of the matching point and Eqs. (134) and (135) will solve for K and S_s. The average thickness of the aquifer will be estimated from the relation

$$b \approx 0.25[2l + l' + d' + 4.48r\sqrt{(1/u_d)}]$$

which is a rearrangement of the time criterion following Eq. (135) and where $1/u_d$ is the recorded value of $1/u$ at the departure point. Having obtained K, S_s, and b, the values of T and S follow.

An example of application is shown in Fig. 29. The field data are from a test in a water-table aquifer [53].

This method is best adapted when the semilogarithmic data curve exhibits curve inflections of the type shown in Fig. 11*a* curve 2, and Fig. 11b, curves 1 and 2, and when the number and distribution of the observed points are such that the curve prior to the attainment of the ultimate straight-line variation is clearly discernable.

2. *Straight-Line Methods*

a. JACOB'S METHOD. Jacob, after whom the method is named, observed that for relatively large values of time ($t/r^2 > 5S/T$), Eq. (132) may be closely approximated [see approximation of $W(u)$] by

$$s = (2.3Q/4\pi T)\log_{10}(2.25Tt/Sr^2) \tag{136}$$

That is, a semilogarithmic plot of s (s being on the uniform scale) versus r^2/t (or t, if one well is used) will, for ($t/r^2 > 5T/S$), be a straight line having a slope $\Delta s/\Delta \log_{10}(r^2/t)$, with an absolute value m equal to $2.3Q/4\pi T$ and an (r^2/t)-intercept, designated by $(r^2/t)_0$ equal to $2.25T/S$; the slope m may be conveniently taken as Δs/one cycle. Consequently, if the best-fit straight line is constructed through the points that appear to define straight-line variation on the semilogarithmic data plot, the field values of T and S may be computed from

$$T = 2.3Q/4\pi m \qquad \text{and} \qquad S = 2.25T/(r^2/t)_0$$

respectively. If the intercept $(r^2/t)_0$ falls outside the data sheet, the coefficient of storage S can be calculated from Eq. (136) and the coordinates of any point on the straight line; T, of course, has been calculated using the slope of the line.

Data collected from the pumped well may similarly be used to determine T. The storage coefficient S cannot, however, be obtained unless the magnitude of the well-loss is known.

b. HANTUSH PARTIAL PENETRATION STRAIGHT-LINE METHOD. For relatively large values of time ($t > bS/2K$), the average drawdown in observation wells located around a partially penetrating well can be approximated by Eqs. (80) and (81), whichever applies, with $a = 1$ and $z = 0.5(l' + d')$ if the aquifer is isotropic. These equations differ from the Theis formula and consequently from Eq. (136) by an additive constant independent of time. Clearly, Jacobs procedure may be followed if the numerical value of this constant can be obtained. If the aquifer thickness is known, this constant (f_s or f_s') can be calculated from the expressions of f_s or f_s' whichever applies [see Eqs. (80) and (81) for definitions of f_s and f_s']. Thus if the semilogarithmic data plot of $\bar{s}$ versus t clearly indicates the formation of the ultimate straight-line variation (Fig. 11, curve 2) (the curve inflection prior to the formation of this line may or may not be apparent), the values of T and S may be calculated [53], respectively, from

$$T = 2.30Q/4\pi m \qquad \text{and} \qquad S = (2.25T/r^2)t_p \exp(f_s)$$

in which t_p is the time-intercept of the ultimate straight line on the zero drawdown axis and f_s is calculated from its defining expression [see Eq. (80)]; if Eq. (81) is the governing equation, f_s' should replace f_s. Calculations for f_s are not as difficult as it may at first appear. Generally only a few terms of the series involved may be required.

c. THEIS RECOVERY METHOD. The residual drawdown in observation wells at relatively large values of time ($t' > 5Sr^2/T$) since shutting down a completely penetrating steadily discharging well may be approximated by Eq. (68). Obviously, a straight line fitted to the points exhibiting straight line variation on the semilogarithmic data plot of s' (s' being on the uniform scale) versus $(1 + t_0/t')$, where t_0 is the period of continuous pumping and t' is the time since shutdown, will have a slope $\partial s'/\partial \log_{10}(1 + t_0/t')$, whose absolute value m (conveniently taken as Δs/one cycle) satisfies the relation

$$T = 2.3Q/4\pi m$$

and thus, makes possible a calculation for T. The storage coefficient cannot be obtained from such data.

For analyzing recovery data around partially penetrating wells, the reader is referred to [53].

B. Methods of Analyses in Leaky Artesian Aquifers

1. Steady-State Methods

If data are collected from three or more observation wells after the flow toward a pumping (or flowing) well has attained essential stability within the region of observation, the following methods may be used to determine field values of the hydraulic properties of the leaky artesian aquifer.

a. JACOB'S TYPE-CURVE METHOD. The procedure of solution is essentially that of the Theis type-curve method. In the present case, however, the type curve [see Eq. (55)] is a logarithmic plot of the function $K_0(x)$ against x and the data plot is that of s versus r; the expression relating x and r is $x = r/B = r\sqrt{(K'/b')/T}$. Consequently, the dual coordinates of the matching point will be used to calculate for T and K'/b' from the relations

$$T = (Q/2\pi)[K_0(x)/s] \qquad \text{and} \qquad K'/b' = T(x/r)^2$$

respectively.

b. STRAIGHT-LINE METHOD. If $r/B < 0.05$, Eq. (55) may be approximated by

$$s = (2.3Q/2\pi T) \log_{10}(1.12B/r),$$

that is, a semilogarithmic data plot of s versus r, s being on the uniform scale, will define a straight-line variation if the observed points fall in the range $r/B < 0.05$. In the range $r/B > 0.05$, the points fall on a curve that approaches the zero drawdown axis asymptotically. A line fitted through the points that appear to define a straight-line variation (three or more points are necessary to establish this condition) will have a slope $\Delta s/\Delta \log_{10} r$ with an absolute magnitude m equal to $2.3Q/2\pi T$ and an r-intercept, r_0, on the zero drawdown axis equal to $1.12B$. Thus T and K'/b' may be calculated from

$$T = 2.3Q/2\pi m \qquad \text{and} \qquad K'/b' = T(1.12/r_0)^2$$

respectively,

2. *Unsteady-State Methods*

These methods are based on the Hantush-Jacob formula for leaky aquifers without storage in the semipervious layers; namely,

$$s = (Q/4\pi T)W(u, r/B) \tag{137}$$

with

$$u = r^2S/4Tt \qquad \text{and} \qquad B = \sqrt{T/(K'/b')} \tag{138}$$

Alternatively, Eq. (137) can be written as

$$s = (Q/4\pi T)[2K_0(r/B) - W(q, r/B)] \tag{139}$$

with

$$q = Tt/SB^2 \tag{140}$$

If $q > 2r/B$, Eq. (139) can, for all practical purposes, be approximated by

$$s_m - s = (Q/4\pi T)W(q) \tag{141}$$

with

$$s_m = (Q/2\pi T)K_0(r/B) \tag{142}$$

where s_m is the maximum or steady-state drawdown.

a. WALTON'S TYPE-CURVE METHOD. This method [56] uses a family of type curves. These type curves are constructed by plotting on a logarithmic paper the function $W(u, x)$ versus $(1/u)$ with x as the running parameter of the family of curves. The curve whose x-value is zero represents the "Theis type curve." The relation between x and r is $r = Bx$. The data plot is that of s versus t/r^2 (or t if one well is used). The procedure of solution is essentially that of Theis. In the matching position, each of the data curves will follow one of the family of the type curves. Because the leakage effects may be insignificant during the early period of pumping, the early points of each of the data curves may fall on or, at least, approach asymptotically the Theis type curve. This type curve will, therefore, serve as a guide in obtaining the best fitting position. The data that may fall on the Theis type curve are generally those points which fall within the period $t < 0.25t_i$ on the semilogarithmic data plot, where t_i is the time at which the inflection of this curve occurs; the position of inflection may be reasonably estimated by inspection or, if the maximum drawdown can be extrapolated from the semilogarithmic data curve, the inflection occurs (see Hantush inflection-point method, presented subsequently) at value s, designated by s_i, given by $s_i = 0.5s_m$.

Having obtained the best matching position, the values of the parameter x corresponding to each of the data curves are obtained by interpolating the position of these curves among the family of type curves. The values of the leakage factor, B, are then computed from the relation $B = r/x$; the mean of these values approximates the average value of B, presumably a constant for a given leaky aquifer. The dual coordinates of the matching point and Eqs. (137) and (138) will solve for T and S. Knowing T and B, the value of K'/b' follows.

Unless a sufficient number of the observed points falls within the period during which leakage effects are insignificant ($t < 0.25t_i$), a unique fitting position is difficult to obtain.

b. HANTUSH TYPE-CURVE METHOD. If the period of test is long enough that a sufficient number of the observed data fall within the period $t > 4t_i$ and that the distribution of these data on the semilogarithmic plot is such that the maximum drawdown can be reasonably extrapolated, a type-curve method based on Eqs. (140) to (142) may be used to estimate numerical values for the hydraulic properties of the aquifer. In this method, the type curve is a logarithmic plot of $W(q)$ versus q (an adequate range of q is $10^{-3} < q < 5$) and the data plot is that of $(s_m - s)$ versus t of all the available observation wells, remembering that the s_m and t_i are different for different well positions. The values of s_m are extrapolated from the semilogarithmic plot of each well and the corresponding values of t_i are estimated by locating the position of the curve inflection, using the values of s_i obtained from $s_i = 0.5s_m$.

In the process of obtaining the best matching position, one should remember that the observed points within the period $t < 4t_i$, for each of the wells, may fall below the type curve, since within this period Eq. (141) may not apply. Actually, within this period, the observed points of each well will follow one of the family of type curves of $W(q, r/B)$ versus q with r/B as the running parameter, had the superposition been made on such a family of curves.

The dual coordinates of the matching point and Eqs. (140) and (141) will solve for T and SB^2. The value of B for each well is then obtained from Eq. (142) and a table of $K_0(r/B)$. The mean of these values will be the average value of the leakage factor, B. Consequently, the average values of S and K'/b' follow.

c. HANTUSH INFLECTION-POINT METHODS. A semilogarithmic plot of s versus t, s being on the uniform scale, has an inflection at which the following relations hold:

$$u_i = r^2S/4Tt_i = r/2B \tag{143}$$

$$m_i = (\partial s/\partial \log_{10}t)_i = (2.3Q/4\pi T)\exp(-r/B) \tag{144}$$

$$s_i = 0.5s_m = (Q/4\pi T)K_0(r/B) \tag{145}$$

$$f(r/B) = \exp(r/B)K_0(r/B) = 2.3s_i/m_i \tag{146}$$

in which s_i, m_i, u_i, and t_i are, respectively, the drawdown, the slope of the semilogarithmic curve, the value of u, and the value of t occurring at the inflection point; s_m is the maximum, or the steady-state drawdown; and $f(r/B)$ is a function for which tabular values are available [28, 57]; it can be readily tabulated by using tables of the exponential function and the zero-order modified Bessel function of the second kind.

For $r/B < 0.01$, that is, for $2.3s_i/m_i$ greater than 4.77, Eq. (146) may be approximated by

$$\log_{10}(2B/r) = 0.251 + s_i/m_i \tag{147}$$

Based on the preceding properties of the semilogarithmic curve (detailed account of which is given in reference [28]), the following methods are developed for obtaining field values of the formation coefficients.

One observation well. If the period of the test is long enough and the number and distribution of the collected data are such that the inflection of the semilogarithmic data curve is apparent and that the maximum drawdown can be reasonably extrapolated, the procedure is as follows:

(1) Extrapolate the maximum drawdown s_m, whence, $s_i = 0.5s_m$.

(2) This value of s_i locates the inflection point on the semilogarithmic data curve, and hence the value of t_i.

(3) Measure the slope m_i of the curve at the inflection point. Usually, this can be closely approximated by the slope of the straight portion of the curve on which the inflection point lies.

(4) Calculate the quantity $2.3s_i/m_i$ and obtain the corresponding value of r/B from a table of $f(r/B)$ or from a graph constructed from this table.

If $2.3s_i/m_i > 4.77$, obtain B/r from Eq. (147). Knowing r, the value of B follows. The values of T, S, and K'/b' are successively computed from Eq. (144) or (145), from Eq. (143), and from $K'/b' = T/B^2$.

Sometimes the extrapolated value of s_m is either over- or underestimated, or/and the straight portion of the data curve (which sometimes may be passed through the observed in several ways) is constructed either flatter or steeper than necessary. A check for the reliability of the calculated formation coefficients should, therefore, be made. This is done by comparing the data

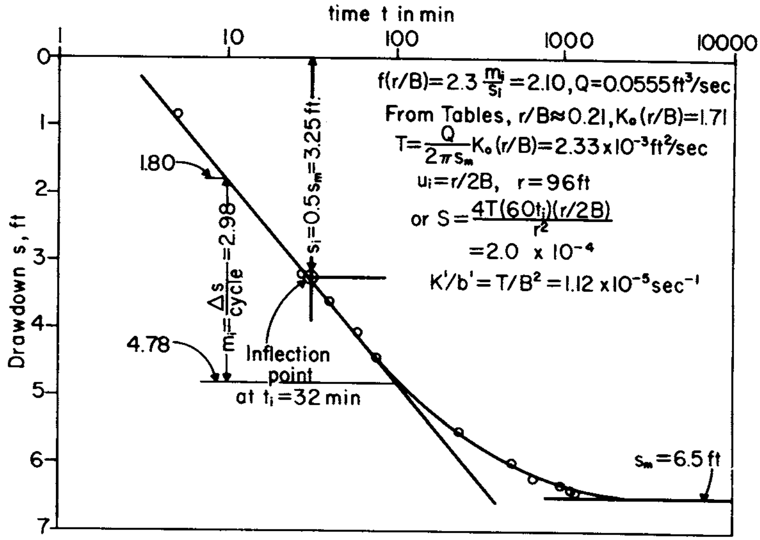

FIG. 30. Example of Hantush inflection-point method for leaky aquifers.

curve with that calculated from Eq. (137) using the determined values of the formation coefficients. The latter part of the collected data is more dependable in the process of comparison, since, as remarked previously (see Theis method), the early part of the data may not conform with theory. If the calculated curve deviates appreciably from that observed (generally, that in the period $t > t_i$), the next step is to adjust the extrapolation of s_m and/or m_i and repeat the analysis. The amount of adjustment is inferred from the amounts and direction of the deviations of the calculated curve from that observed. With a little experience, only one trial may result in a satisfactory answer.

Figure 30 is an example of application. This example has been analyzed by Walton's method [56] also.

More than one observation well. Equation (144) may be written as

$$r = 2.3B[\log_{10}(2.3Q/4\pi T) - \log_{10}m_i] \tag{148}$$

Clearly, a semilogarithmic plot of r against m_i (r being on the uniform scale) should be a straight line having a slope $\Delta r/\Delta \log_{10} m_i$ equal to $2.3B$ and an m_i-intercept [designated by $(m_i)_0$] equal to $2.3Q/4\pi T$. Thus, if the period of a test is long enough that the straight portions of the semilogarithmic time-drawdown curves of at least two observation wells are fully developed, the following procedure may be used to calculate the formation coefficients:

(1) Find the slope m_i of the straight portion of each semilogarithmic data curve.

(2) Construct the best-fit straight line on the semilogarithmic (m_i versus r) plot (r being on the uniform scale) and obtain its slope $\Delta r/\Delta \log_{10} m_i$ and its m_i-intercept $(m_i)_0$.

(3) Calculate B from $B = 0.434(\Delta r/\Delta \log_{10} m_i)$, and T from $T = 2.3Q/4\pi(m_i)_0$ or from Eq. (144) and the coordinates of any point on (m_i versus r)-straight line.

(4) Calculate K'/b' from $K'/b' = T/B^2$.

(5) With this information, compute the values of s_i for each well by using Eq. (145) and a table of $K_0(r/B)$, then locate on each of the data curves the corresponding inflection point; hence obtain the values of t_i.

(6) Calculate the corresponding values of S by using Eq. (143). The mean of these values approximates the average value of S in the well field.

C. Analyses of Pumping Tests in Water-Table Aquifers

1. *Methods of Analyses in Horizontal Aquifers*

The methods of analyses for tests in artesian aquifers can be applied to analyze corresponding situations of complete and partial penetration in water-table aquifers if $s/D_0 < 0.25$ (D_0 being the initial depth of saturation) and provided the quantities s, T, and S in the former equations are replaced with $s - s^2/2D_0$, KD_0, and ϵ (specific yield), respectively, (Section V, B, 2); for partial-penetration equations for relatively short time or relatively thick aquifers, s is replaced with $s - s^2/2l$, l being the penetration depth of the pumped well (Section V, B, 3). Consequently, the methods of analyses, previously presented, may be used in analyzing data collected from pumping tests in water-table aquifers if the foregoing adjustments are made. It is to be observed, however, that the early part of the data may not conform with theory, since the theory neglects the effects on the nature of the flow of the vertical-flow components, of the time-lag in the release of stored water, and of the quantity of water released from storage due to aquifer compaction, which effects may be appreciable in the early stages of the flow. Eventually, however, these effects are minimized.

2. *Methods of Analyses in Sloping Nonleaky Water-Table Aquifers*

From Eq. (94), the drawdown around a well steadily discharging from an effectively infinite and sloping water-table aquifer (see Fig. 15, for coordinate system) can be approximated by

$$(s-s^2/2D_0) = (Q/4\pi T_0)\exp[-(r/\beta)\cos\theta]W(u, r/\beta) \tag{149}$$

with

$$T_0 = KD_0, \qquad u = r^2\epsilon/4T_0t, \qquad \text{and} \qquad \beta = 2D_0/i \tag{150}$$

The corresponding equation for the maximum drawdown s_m, or the drawdown during the steady state, is given by

$$(s_m - s_m^2/2D_0) = (Q/2\pi T_0)\exp[-(r/\beta)\cos\theta]K_0(r/\beta) \tag{151}$$

For $q > 2r/\beta$, Eq. (149) may be approximated by

$$(s_m - s_m^2/2D_0) - (s-s^2/2D_0) = (Q/4\pi T_0)\exp[-(r/\beta)\cos\theta]W(q) \tag{152}$$

with

$$q = T_0t/\epsilon\beta^2 \tag{153}$$

Equation (137) becomes identical to Eq. (149) if the quantities, s, $Q/4\pi T$, T, S, and B, of the former aquation are replaced with $s-s^2/2D_0$, $(Q/4\pi T_0) \cdot \exp[-(r/\beta)\cos\theta]$, T_0, ϵ, and β, respectively. Similarly, Eqs. (138), (142), (141) and (140) become identical to Eqs. (150) to (153), respectively. This analogy suggests the possibility of applying, to the present flow system, the graphical procedures outlined for the leaky artesian system.

a. TYPE-CURVE METHODS. For a given aquifer, the Walton and the Hantush type-curve methods for leaky aquifers can be used within the same range of applicability; it is assumed that the orientation with the direction of natural flow of the line joining the pumped and observed well is known (θ is known).

When applying Walton's method to the present flow system, the family of type curves is the same as previously used, but the data logarithmic plot is that of $s-s^2/2D_0$ versus t. The dual coordinates of the matching point and Eqs. (149) and (150) determines the quantities ϵ/T_0 and $(Q/4\pi T_0)\exp[-(r/\beta)\cos\theta]$. The position of the data curve determines r/β and, hence β. Simultaneous solutions of these data determine the values for T_0 and ϵ. The slope of the tilted aquifer is then obtained from $i = 2D_0/\beta$.

When applying Hantush method to the present flow system, the type curve is the same, but the data curve is that of $[(s_m - s_m^2/2D_0) - (s-s^2/2D_0)]$ versus t, the value $s_m - s_m^2/2D_0$ being extrapolated from the semilogarithmic plot of the observed values of $(s-s^2/2D_0)$ versus time. The dual coordinates of the matching point and Eqs. (152) and (153) determine the quantities $\epsilon\beta^2/T_0$ and $(Q/4\pi T_0)\exp[-(r/\beta)\cos\theta]$. Equation (151), the second of these two quantities and a table of $K_0(r/\beta)$, determine r/β and, hence β. Simultaneous solutions of these data determine T_0, ϵ, and i.

These two methods apply to each well individually. They cannot be applied to a composite data plot, since $\log_{10}[(Q/\Delta\pi T_0)\exp[-(r/\beta)\cos\theta]]$ is not a constant for variable r and θ.

b. INFLECTION-POINT METHODS. The properties of a semilogarithmic plot of $s-s^2/2D_0$ versus t (t being on the logarithmic scale) are given by Eqs. (143) to (146) if s_i, s_m, $Q/4\pi T$, T, S, and B are replaced, respectively, with $s_i-s_i^2/2D_0$, $s_m-s_m^2/2D_0$, $(Q/4\pi T_0)\cdot\exp[-(r/\beta)\cos\theta]$, T_0, ϵ, and β. Consequently, if these replacements are observed in the procedure previously outlined for the Hantush inflection-point method for *one observation well* in a leaky artesian aquifer, calculations for T_0, ϵ, β, and i may be similarly effected. Equation (147) cannot be used, however, in the present calculations.

If the replacements discussed in the preceding paragraph are observed in the procedure outlined for the Hantush inflection-point method for *more than one well* in a leaky artesian aquifer, and observing that the slope $\Delta r/\Delta\log_{10}(m_i)_0$ of the semilogarithmic (m_i versus r) plot will be equal to $2.3\beta(1+\cos\theta)$ instead of $2.3B$, calculations for T_0, ϵ, β, and i can be similarly performed, provided that the observation wells lie on a radial line whose inclination to the direction of the natural flow is known; that is, θ has the same value for all the wells.

D. Methods of Analyses in Wedge-Shaped Nonleaky Artesian Aquifers

The drawdown distribution around wells completely penetrating and pumping or flowing from a nonleaky wedge-shaped artesian aquifer whose thickness may, for all practical purposes, be assumed to vary exponentially in the direction of x while remaining constant in the y direction has been obtained by Hantush [29]. For the case of a steadily discharging well, the equation of drawdown is analogous to that for the drawdown around wells steadily discharging from a sloping nonleaky water-table aquifer, that is, to Eq. (149). In fact, the former equation can be obtained from Eq. (149) by merely replacing herein the quantities $s-s^2/2D_0$, D_0, T_0, ϵ, and β with s, b_0, Kb_0, b_0S_s, and $(-a)$, respectively, where b_0 is the thickness of the wedge-shaped aquifer at the site of the pumping well and a is a constant defining the variation of this aquifer thickness (see reference [29] for detailed derivation, coordinate axis, and limitation of use of the drawdown equation). In the process of replacement, one should recall that the functions $W(u,x)$ and $K_0(x)$ in the developed equations represent the functions $W(u,|x|)$ and $K_0(|x|)$ respectively. Consequently, if these replacements are observed in carrying out the procedures outlined for the methods of analyzing data from tests in sloping nonleaky water-table aquifers, calculations for K, S_s, and a—the parameter of thickness variation—may be made.

E. Methods for Determining Well Characteristics

1. Drawdown in Discharging Wells

The drawdown in a discharging well is made up of the head loss resulting from the turbulent flow of water into and inside the well, referred to as *well loss*, and from the *formation loss* resulting from the head loss associated with turbulent flow near the well and with laminar flow in the remainder of the aquifer.

Well development and gravel packing may increase the permeability of the formation next to the well with the consequence of lessening the drawdown in and near the well. Because of the difficulty of expressing analytically the effects on the drawdown of the turbulent flow and the increase of permeability in the formation near the well, Jacob [54] suggested that these effects may be accounted for approximately by introducing the concept of the *effective well radius*. This hypothetical, empirically determined radius if substituted in the drawdown equation of that well, will yield the actual drawdown outside the screen of an artesian well. Similarly this radius will yield the actual water level in a well under water-table conditions if well losses are negligible. According to this assumption, the effective radius may be greater or smaller than the actual radius of the well depending on whether the decrease in drawdown caused by the permeability improvement is greater or smaller than the amount of head loss associated with turbulence near the well. Jacob expressed the well loss as CQ^2, where Q is the discharge of the well and C is a constant, referred to as the *well-loss constant*. Jacob's assumptions express the drawdown, s_0, in the well as

$$s_0 = s_w(r_w, t) + CQ^2$$

or

$$s_0 = B(r_w, t)Q + CQ^2 \tag{154}$$

where $B = s_w/Q$ is the head loss in the formation per unit discharge and s_w is the drawdown as given by the drawdown equation for the well, with $r = r_w$.

In his treatment of the subject, Rorabaugh [55] suggested that the head loss associated with turbulence in the formation as well as into and inside the well be expressed as CQ^n where n is a constant, greater than 1 and may exceed 2, to be determined for individual wells. Accordingly, he expressed the drawdown in the well as

$$s_0 = BQ + CQ^n \tag{155}$$

Observations [55] have shown, however, that prediction of drawdowns within the range of operating well discharges as computed from Jacob's equation and from that of Rorabaugh's, agreed very closely with observed data, although such prediction differed appreciably for discharges greatly in excess of those used in the test. Rorabaugh [55] suggested that, if prediction must be made for large discharges (for the purpose of pump design or for determination of maximum yield), a margin of safety can be provided by computations by both methods, using the least favorable solution for the purpose. He also observed that results obtained from his equation are not applicable to problems of head distribution outside the well or for designing radii of wells, and that Jacob's equation describes more nearly the true head distribution outside the well and should be used for this type of problem.

2. *Specific Capacity of Wells*

The *specific capacity*, Sp.C., of a well is defined as the ratio of its discharge to its drawdown. By using Jacob's expression for the drawdown in the well, the specific capacity will be given by

$$\text{Sp.C.} = Q/s_0 = 1/[s_w(r_w, t)/Q + CQ] \tag{156}$$

Showing that the specific capacity decreases with time and discharge. Obviously, the practice of assuming that the discharge is directly proportional to drawdown, implying a constant specific capacity, may introduce sizable errors. The influence of the well radius on the specific capacity depends on the magnitude of the discharge Q as well as on the empirical constant C which in turn depends upon the well radius among other factors. Clearly, the radius of the well is of importance in well design, especially if the design discharges are relatively high. If, however, the head losses associated with turbulence near, into, and inside the well are relatively small, the influence of the well radius on the specific capacity may be relatively unimportant, since, in this instance, Sp.C. varies with $\ln r_w$, a term that varies little as r_w is changed.

The constants r_w and C for a given well, being empirical, their determination must depend on observation. The values of r_w and C could be determined by running three or more drawdown tests at different rates of discharge, with periods of rest in between of sufficient duration to ensure nearly complete recovery. Drawdowns taken at fixed intervals of time from the beginning of each test may be used in conjunction with Eq. (154) to solve graphically for B and C, or with Eq. (155) for obtaining B, C, and n [6]. These constants can be obtained somewhat more conveniently by what is referred to as a step-drawdown test.

3. *Step-Drawdown Test in Artesian Aquifers*

The values of r_w, C, and n for a given well may be determined [54] by a step-drawdown test in which the well is operated during successive periods at constant fraction of full capacity (Fig. 31). In addition, observations must be made in outlying wells to determine S, if a value of this constant is not available; otherwise r_w cannot be found.

By applying the principle of superposition to Eq. (154) (Jacob's equation), the drawdown in the well, s_{0m} during the mth step may be written as

$$s_{0m}(t) = CQ_m^2 + \sum_{i=1}^{m} (Q_i - Q_{i-1})B(r_w, t - t_i)$$

in which Q_m is the discharge during the mth step, Q_i is the discharge during the ith step of those preceding the mth one, with $Q_0 = 0$, and t_i is the time at which the ith step began, with $t_1 = 0$.

The sum of the increments of the drawdown taken at a fixed interval of time from the beginning of each step (that is, at $t - t_i = t^*$) may be obtained from the preceding equation as

$$\sum_{i=1}^{m} \delta s_{0i}(t^*) = s_0(t^*)_m = B(r_w, t^*)Q_m + CQ_m^2 \tag{157}$$

in which $\delta s_{0i}(t^*)$ is the drawdown increment between the ith step and that preceding it taken at time t^* from the beginning of the ith step, $s_0(t^*)_m$ is the drawdown in the well that would have taken place at $t = t^*$, had the well been pumped at the constant rate Q_m, and $B(r_w, t^*)$ is the head loss in the formation per unit discharge at time t^* since the commencement of pumping.

If Rorabaugh's expression replaces Jacob's, the equation corresponding to Eq. (157) will be

$$s_0(t^*)_m = BQ_m + CQ_m^n$$

a. DETERMINATION OF THE COEFFICIENTS $B(r_w, t^*)$ AND C. Based on Eq. (157), data collected from at least three steps of a step-drawdown test may be used to calculate for B and C as follows:

(1) Obtain the increments of drawdown $\delta s_{0i}(t^*)$, with t^* usually taken as 1 or 2 hr. $\delta s_{0i}(t^*)$ is determined in each step by taking the difference between the observed water level at time t^* since the beginning of the step and the corresponding water level on the extension of the preceding drawdown curve. In performing this extension (the extrapolation of time-drawdown variation), one should be guided by the general trend of the theoretical semilogarithmic time-drawdown variation around wells operating in the flow system under consideration.

(2) Obtain $s_0(t^*)_m$ corresponding to the discharge Q_m from the relation $s_0(t^*)_m = \delta s_{01}(t^*) + \delta s_{02}(t^*) + \cdots + \delta s_{0m}(t^*)$, then compute the ratio $s_0(t^*)_m/Q_m$.

(3) Plot Q_m versus $s_0(t^*)_m/Q_m$ on uniform scales and obtain the best-fit straight line.

(4) The slope of this line $[\Delta(s_0(t^*)_m/Q_m)/\Delta Q_m]$ is equal to C.

(5) Equation (157) and the coordinates of any point on this line will solve for $B(r_w, t^*)$.

An example of analysis of data from a well completely penetrating a nonleaky artesian aquifer is given in Fig. 31. Should it become necessary to base the analysis on Rorabaugh's assumptions, (as when prediction of drawdowns for high ratio of discharges are desired) the procedure of analysis is essentially the same, except that the values of $[s_0(t^*)/Q-B]$ will be plotted versus Q on

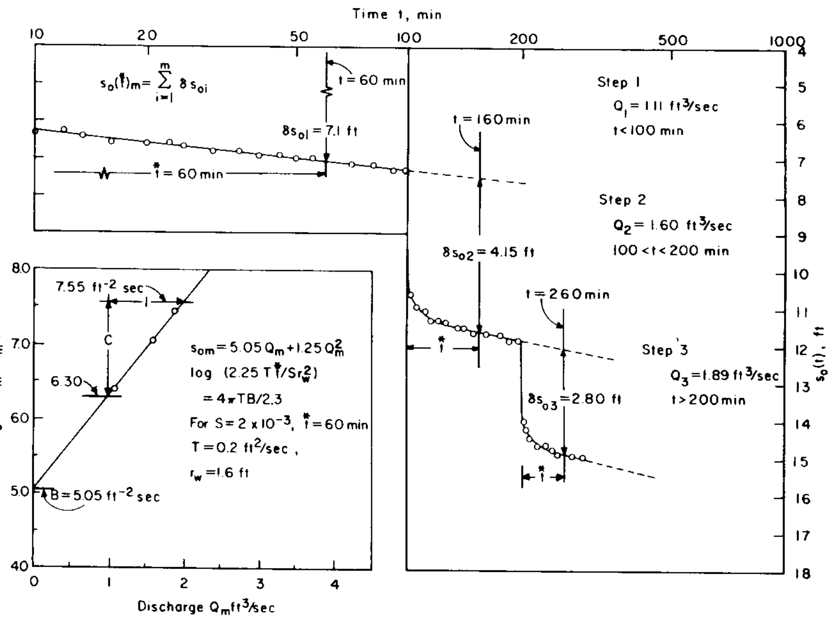

FIG. 31. Example of step-drawdown test.

logarithmic paper for assumed values of B. The value of B that gives the straightest line on this plot will be the required one, the intercept of the line on the axis $Q = 1$ is equal to the value of C, and the slope $[\Delta(s_{0m}/Q-B)/\Delta Q]$ is equal to $n-1$ from which the value of n is obtained.

b. DETERMINATION OF THE EFFECTIVE RADIUS r_w. Having determined B from analysis based on Jacob's assumptions and having obtained T and S from other tests or from observations in outlying wells during the step-drawdown test, the value of r_w may be obtained by substituting these data in the appropriate expression for $B(r_w, t^*)$ and solving for r_w. T may be obtained also from analyzing data collected during the first step of the step-drawdown test, using the method of analysis applicable to the flow system under consideration. For example:

(1) Artesian wells completely penetrating nonleaky aquifers. The expression

for B for such wells, from Eq. (63) and the definition of $B[B = s_w(r_w, t)/Q]$, provided $t > (30r_w^2S/T)$, may be written as

$$W(r_w^2S/4Tt^*) = 4\pi TB \tag{158}$$

Consequently, a table of $W(u)$ and the data from the test will produce a solution for r_w. In general, however, t^* (1 or 2 hr) satisfies the relation $t^* > 30r_w^2S/T$ (the nominal radius of the well may be used to determine this criterion), in which case $W(u)$ may be replaced by its approximate form for relatively large times; hence r_w may be computed (see Fig. 31) from

$$\log_{10}(2.25Tt^*/Sr_w^2) = 4\pi TB/2.3 \tag{159}$$

If $t^* < 30r_w^2S/T$ (the nominal radius of the well may be used for checking this criterion), the expression for B is that obtained from Eq. (64); consequently r_w is computed from

$$S(Tt^*/Sr_w^2, 1) = 4\pi TB \tag{160}$$

by using a table of $S(\tau,\rho)$, namely Table VI.

(2) Artesian wells partially penetrating nonleaky aquifers. If $t^* > 30r_w^2S/T$, the expression for $B(r_w, t^*)$ for the case of a well screened throughout its depth of penetration in a nonleaky aquifer is given by Eq. (78), with $B = \infty$ and $K_z = K_r = K$, provided $l/r_w > 10$. After some manipulation, the expression may be written as

$$\log_{10}(2b/r_w) = (4\pi TB + 0.846)/4.6m - (1/2m)\log_{10}Ag \tag{161}$$

where

$$m = b/l, \qquad A = 0.14Tt^*/Sb^2, \qquad g = [(2m+1)/m(2m-1)]^{2m}$$

from which r_w can be readily obtained.

4. *Values of r_w and C for Water-Table Wells*

The analysis leading to Eq. (157) holds approximately for water-table conditions, if $s_0/D_0 < 0.02$, D_0 being the initial depth of flow. Consequently, data from a step-drawdown test in water-table wells may be similarly analyzed only if $s_0/D_0 < 0.02$.

For $t > 5\epsilon D_0/K$ and provided $D_0 - h_w < 0.5D_0$, the drawdown in a well operating in a nonleaky water-table aquifer, from Eq. (91) and Jacob's well loss equation, can be approximated by

$$s_0 = CQ^2 + D_0 - [D_0^2 - (Q/2\pi K)\ln(2.25T_0t/\epsilon r_w^2)]^{0.5}$$

which can be rewritten as

$$y = 1 - B'(r_w,t)Q \tag{162}$$

where

$$y = [1 - s_0/D_0 + CQ^2/D_0]^2 \tag{163}$$

and

$$B' = (2.3/2\pi KD_0^2)\log_{10}(2.25\,T_0 t/\epsilon r_w^2) \tag{164}$$

The values of r_w and C for a given well may be determined by running three or more drawdown tests at different constant rates of discharge, with periods of rest in between of sufficient duration to insure nearly complete recovery. If values of y (computed from assumed values of C and observed drawdowns at a fixed time t^*, usually 1 or 2 hr since the beginning of pumping) are plotted versus Q on uniform-scale paper, the required value of C will be that which will give the straightest line. The absolute value of the slope of this line $\Delta y/\Delta Q$ is equal to $B'(r_w, t^*)$. This value of B', the known values of T_0 and ϵ, and Eq. (164) will produce a solution for r_w.

Provided $t^* > 30r_w^2\,\epsilon/T_0$ and $l/r_w > 10$, the constant r_w for a partially penetrating well, that is screened throughout its depth of penetration l in an infinite nonleaky aquifer, may be computed from the relation

$$\log_{10}(2D_0/r_w) = [2\pi KD_0^2 B'(r_w, t^*) + 0.846]/4.6m - (1/2m)\log_{10}\mathrm{Ag} \tag{165}$$

where

$$m = D_0/l, \qquad \mathrm{A} = 0.14T_0 t^*/\epsilon D_0^2, \qquad g = [(2m+1)/m(2m-1)]^{2m}$$

F. Other Methods

The methods presented in the previous articles pertain to constant-discharge wells in aquifers of infinite areal extent. Other methods pertaining to other flow conditions and/or other flow systems are available. Among these are the following:

For wells in infinite aquifers, there are Jaeger method [58], Chow method [5, 59], Ferris slug method [52], Skibitzke bailer method [52], Stallman varying discharge method [52], Jacob-Lohman constant-head method [16, 52], Brown-Theis intermittent discharge method [52, 60], Hantush partial-penetration recovery method [53], and Abu Zied-Scott exponentially decreasing discharge method [62].

For wells near a hydraulic boundary, there are Kazmann method [42], Rorabaugh method [43], Hantush methods [41], Stallman method [52], and Moulder method [52].

Major Symbols

$A(\tau,\rho)$ Flowing well function (see Section II, C, 1)

B Leakage factor $=\sqrt{Kb/(K'/b')}$ for artesian aquifers, L
$=\sqrt{K\bar{b}/(K'/b')}$ for water-table aquifers, L
$=(S_w/Q)$ The formation loss per unit discharge, T/L^2

B_z, B_r Values of B after replacing K with K_z and K_r, respectively, L

b Uniform thickness of artesian aquifers; also initial depth of flow around collector wells in water-table aquifers, L

b' Uniform thickness of a semipervious layer of leaky systems, L

$\bar{b} = 0.5\ (D_{0m} + \bar{D}_m)$
$\approx 0.5[D_{0w}(\tau_0) + D_{0w}(\tau_0 + t_0)]$ for distances $> 1.5D_0$
$\approx 0.5\ [D_w(0) + D_w(t_0)]$ for distances $< 1.5D_0$

D Height of water-table above the base of the aquifer, L

D_w Depth of saturation just outside the screen of a water-table well, L

$D_w(t)$ Value of D_w at any time t,L

$D_0(r,\theta,\tau_0 + t)$ The distribution of the depth of flow that would prevail in a water-table aquifer if the well were not pumped, L

D_{0m} Weighted mean of the areal distribution of the depth of flow during the period of induced flow if the well were not pumped, L

$D_{0w}(\tau_0 + t)$ Value of D_0 at the face of the well at any time, L

$\bar{D}$ Depth of water in an idle water-table well screened throughout the aquifer, L

$\bar{D}_m$ Weighted mean of the areal distribution of the depth of flow during the period of induced flow, L

erf(x), erfc(x) Error and the complementary error functions (see Section II, C, 2)

$G(\tau)$, $G(\tau,\beta)$ Flowing well discharge functions for nonleaky and leaky aquifers (see Section II, C, 3, 4)

$H(u,\beta)$ An infinite integral (Section II, C, 5)

i^nerfc(x) nth repeated integral of the error function (Section II, C, 7)

i Slope of a tilted aquifer; also an index of summation

K, K' Isotropic hydraulic conductivities in a main aquifer and in a semipervious layer, respectively, L/T

$K'/b' = T/B^2$ Coefficient of leakage, $1/T$

$K_0(x)$, $K_1(x)$ Zero- and first-order modified Bessel functions of the second kind (Section II, C, 9)

$L(u, \pm w)$, $L(u, 0)$ Functions (Section II, C, 10)

$M(u,\beta)$ An infinite integral (Section II, C, 12)

Q Constant discharge of a well, L^3/T

$r = [(x-x_0)^2+(y-y_0)^2]^{0.5}$ Radial distance from center of a pumping well located at (x_0, y_0) to any point in the surrounding area, L

$r_1 = [(x+x_0)^2+(y-y_0)^2]^{0.5}$ Radial distance from the center of the image in the y-axis of a well located at (x_0, y_0), L

r_w Effective radius of a well (Section X, E), L

$S(\tau,\rho)$ Well function for a well of any radius r_w (Section II, C, 13)

S Storage coefficient of an artesian aquifer

S' Storage coefficient of a semipervious layer

$S_s = S/b$ Specific storage of an artesian aquifer, $1/L$

$S'_s = S'/b'$ Specific storage of a semipervious layer, $1/L$

$\sinh^{-1}(x)$ The inverse hyperbolic sine (Section II, C, 14)

$s(r,t)$ Average drawdown in idle wells screened throughout an aquifer; also drawdown of water table in Boulton's equation, L

$s(r,z,t)$ Piezometric drawdown, or drawdown in piezometers open at the point (r, z), L

s_w Drawdown in a discharging well, if well losses are negligible, L

s_0 Total drawdown in a discharging well, L

$T = Kb$ Transmissivity (transmissibility) of artesian aquifers, L^2/T

$T_0 = KD_0$ Transmissivity corresponding to initial depth of flow in a water-table aquifer, L^2/T

t, t' Times since discharging and since shutting down of a well, respectively, T

t_0 Period of continuous pumping or of other types of induced flow, T

$t^* =$ Empirical constants in discharge equations for wells of variable discharge, T

= Time since incidence of deep percolation at which a drainage well begins to pump, T

= Fixed interval of time measured from the beginning of each step of a step-drawdown test, T

$U_0 = x_0/\sqrt{4\nu t}$

$u = r^2/4\nu t$, $u_a = a^2/4\nu t$, $u_1 = r_1^2/4\nu t$, $u_r = r^2/4\nu_r t$, $u_d = r^2/4\nu t_d$, $u_i = r^2/4\nu t_i$, $u' = r^2/4\nu t'$; also rate of vertical leakage into or out of an artesian aquifer, L/T

$V(\tau,\rho)$ Gravity well function (Section II, C, 15)

$W(u)$ Well function for nonleaky aquifers (Section II, C, 16)

$W(u,\beta)$ Well function for leaky aquifers Section II, C, 17)

w Rate of uniform deep percolation, L/T; also rate of vertical leakage into or out of an artesian aquifer, L/T

$\alpha = [1/\beta^2 + 1/B^2]^{0.5}$ Parameter in formulas pertaining to leaky sloping water-table aquifers, $1/L$

$= r\cos(\theta - \theta_i) - r_c$, Parameter in formulas for collector wells, L

$\beta = (r/4B)\sqrt{S'/S}$ Parameter in formulas pertaining to leaky aquifers with storage in semipervious layer

$= 2\bar{b}/i$ Parameter in formulas pertaining to sloping water-table aquifer, L

$= r\sin(\theta - \theta_i)$ Parameter in formulas for collector wells, L

= compressibility of water, L^2/F

$\Gamma(x) =$ the gamma function (Section II, C, 20)

$\gamma = 2(a - r_c)/l$ Parameter in formulas for collector wells; also unit weight of water, F/L^3

$\delta = r\cos(\theta - \theta_i) - l'$ Parameter in formulas for collector wells, L

$\delta_1 = 1 + S'/3S$, $\delta_2 = 1 + S'/S$ } in formulas pertaining to leaky systems

ϵ Specific yield $\approx$ effective porosity

$\mu = (2a - 2r_c - l)/l$ Parameter in formulas for collector wells

$\nu = K\bar{b}/\epsilon$ Parameter in formulas pertaining to flow in water-table aquifers, L^2/T

$\nu = T/S = K/S_s$, $\nu_z = K_z b/S$, $\nu_r = K_r b/S$ Parameters in formulas pertaining to flow in artesian aquifers, L^2/T

$\rho = r/r_w$ Parameter in formulas for flowing wells and in the function $S(\tau,\rho)$

$= r/D_0$ Parameter in formulas for water-table wells

$= r_c/l$ Parameter in formulas for collector wells

$\sigma =$ Compressibility of solid skeleton of an aquifer, L^2/F

$\tau = Kt/\epsilon D_0$, $\tau' = Kt'/\epsilon D_0$ Time factors in formulas for water-table wells

= Time since incidence of deep percolation in problems of drainage wells, T

$= \nu t/r_w^2 = Tt/Sr_w^2$; time factor in formulas for flowing wells

$\tau_0 =$ Time since an arbitrarily chosen reference of time, at which induced flow begins, T

= Period of continuous uniform deep percolation, T.

$\varphi = p/\gamma + z$; piezometric head = hydraulic head, L

$\bar{\varphi} =$ Average head in idle wells screened throughout an aquifer, L

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